SCALAR PRODUCTS

5 minute review. Recap how the standard unit basis vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ work, and cover the fact that $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ is often written as (a, b, c). Recap the scalar product, both as $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos\theta$ (where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$) and as $(a, b, c) \cdot (x, y, z) = ax + by + cz$. Remind students how to find the angle between two vectors (or save it for the warm-up).

Class warm-up. A force $\mathbf{F_1}$ has magnitude 3N and makes angles of $\frac{\pi}{3}$, $\frac{2\pi}{3}$ and $\frac{3\pi}{4}$ with $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ respectively. Express $\mathbf{F_1}$ in component form, and find the angle between $\mathbf{F_1}$ and $\mathbf{F_2} = \mathbf{i} + \mathbf{j} + \mathbf{k}$. (Use a diagram and trigonometry, or solve algebraically.)

Problems. Choose from the below.

1. **Solving vector equations**.

   (a) The vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are given by
   $$\mathbf{a} = (p, 1, -3), \quad \mathbf{b} = (1, q, 1), \quad \mathbf{c} = (0, 1, r)$$
   where p, q, r are unknown. Given that $\mathbf{a} + 2\mathbf{b} = 2\mathbf{c}$, find p, q and r.

   (b) Given that $\mathbf{a} = 2\mathbf{i} + 4\mathbf{j} - 3\mathbf{k}$, $\mathbf{b} = t\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and that $\mathbf{a}$ is perpendicular to $\mathbf{b}$, calculate t.

2. **Forces**.

   (a) A force $\mathbf{F}$ of magnitude 20N is inclined at an angle θ to the horizontal. Given that its vertical component is 10N, find the horizontal component and the value of θ.

   (b) Three forces $\mathbf{F}_1$, $\mathbf{F}_2$ and $\mathbf{F}_3$ act on a particle. Given that $\mathbf{F}_1 = \mathbf{i} + 2\mathbf{k}$, $\mathbf{F}_2 = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$ and that the resultant force is given by $\mathbf{F} = 2\mathbf{i} + \mathbf{j}$, find $\mathbf{F}_3$.

3. **Vectors and geometry**.

   (a) Given the points $A(-2, 3, 5)$, $B(3, 1, 6)$ and $C(13, -3, 8)$, express $\overrightarrow{AB}$ and $\overrightarrow{AC}$ in terms of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$. Show that the points A, B, C lie on a straight line and calculate the ratio $AB : BC$.

   (b) Find the angles (to the nearest degree) of the triangle whose vertices are at $(5, 3, 2)$, $(6, 5, 4)$ and $(7, -1, 3)$.

4. **More geometry**.

   (a) The *distributive law* for the scalar product says, roughly speaking, that brackets can be expanded in the usual way; that is, $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$. Simplify $(\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b})$.

   (b) The points A, B and C lie on a circle centred at the origin O and AOC is a diameter of the circle. Let $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$. Draw a diagram and express $\overrightarrow{BA}$, $\overrightarrow{BC}$ and $\overrightarrow{BA} \cdot \overrightarrow{BC}$ in terms of $\mathbf{a}$ and $\mathbf{b}$. Hence show that angle ABC is a right angle.

For the warm-up, $\mathbf{F_1} = (3/2, -3/2, -3/\sqrt{2})$ and it makes an angle of 1.991 radians with $\mathbf{F_2}$ (to 3d.p.).

Selected answers and hints.

1. (a) $p = -2$, $q = 1/2$, $r = -1/2$.
   (b) $t = -5/2$.
2. (a) The horizontal component is $10\sqrt{3}$N, and $\theta = \pi/6$.
   (b) $F_3 = 3\mathbf{j} - 5\mathbf{k}$.
3. (a) $\overrightarrow{AB} = 5\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\overrightarrow{AC} = 15\mathbf{i} - 6\mathbf{j} + 3\mathbf{k} = 3\overrightarrow{AB}$. The ratio $AB : BC$ is $1 : 2$.
   (b) The angles are 28°, 45° and 107°, to the nearest degree.
4. (a) $(\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b}) = |\mathbf{a}|^2 - |\mathbf{b}|^2$.
   (b) $\overrightarrow{BA} = \mathbf{a} - \mathbf{b}$, $\overrightarrow{BC} = -\mathbf{a} - \mathbf{b}$ and $\overrightarrow{BA} \cdot \overrightarrow{BC} = |\mathbf{b}|^2 - |\mathbf{a}|^2 = 0$, so $\overrightarrow{BA}$ and $\overrightarrow{BC}$ are perpendicular.

For more details, start a thread on the discussion board.