

DE MOIVRE'S THEOREM

5 minute review. Recap de Moivre's Theorem, $\cos(n\theta) + i\sin(n\theta) = (\cos\theta + i\sin\theta)^n$, and how to solve $z^n = r(\cos\theta + i\sin\theta)$ for z , perhaps by doing the warm-up below.

Class warm-up. Find the cube roots of $1 + i$.

Problems. Choose from the below.

- (a) Find the complex solutions of $z^2 + 2z + 4 = 0$ and draw them on an argand diagram.

- (b) Hence find all the solutions to $w^2 = -2 - \frac{4}{w^2}$.

- More trigonometric identities.**

- (a) Use de Moivre's Theorem to show that $\sin(3\theta) = -4\sin^3(\theta) + 3\sin(\theta)$. What is $\cos(3\theta)$ in terms of powers of $\cos\theta$?
- (b) Suppose $\sin\theta = 0$. Deduce that $\sin(\theta/3) = \pm\sqrt{3}/2$ or 0. Find values of θ that give each of these answers.
- (c) Find $\sin(5\theta)$ in terms of powers of $\sin\theta$ and deduce that

$$\sin^2(\pi/5) = (5 \pm \sqrt{5})/8 \text{ or } 0.$$

Which is it?

- Roots of unity.** For a positive integer n , the n th-roots of unity are defined to be the solutions to $z^n = 1$. There are always n such solutions.

- (a) Find the 5 fifth-roots of unity and plot them on the argand plane. Let ω be the solution with the smallest positive argument. What is the effect in the argand plane of multiplying a complex number by ω ?
- (b) Find the 5 fifth-roots of $1 - i\sqrt{3}$ and plot them on the argand plane.
- (c) Show that if ω is as in (a) and z is a fifth-root of $1 - i\sqrt{3}$ then ωz is also a fifth-root of $1 - i\sqrt{3}$. Can you express the other fifth-roots of $1 - i\sqrt{3}$ in terms of z and ω ? (Hint: think about multiplication by ω .)

- More roots of unity.** Let $n > 1$ be a positive integer, and let $\omega = \cos(\frac{2\pi}{n}) + i\sin(\frac{2\pi}{n})$. By considering the expansion of $(z - 1)(z^{n-1} + z^{n-2} + \dots + z + 1)$ show that

$$1 + \omega + \omega^2 + \dots + \omega^{n-1} = 0$$

and hence find a value for $\cos(\frac{2\pi}{n}) + \cos(\frac{4\pi}{n}) + \dots + \cos(\frac{2(n-1)\pi}{n})$.

For the warm-up, $1 + i = \sqrt{2}(\cos(\pi/4) + i \sin(\pi/4))$, so the roots are of the form

$$z_p = \sqrt{2}^{-1/3} \left(\cos \left(\frac{\pi/4 + 2p\pi}{3} \right) + i \sin \left(\frac{\pi/4 + 2p\pi}{3} \right) \right) \quad \text{for } p = 0, 1, 2.$$

In other words, we have

$$\begin{aligned} z_0 &= 2^{1/6} \left(\cos \left(\frac{\pi}{12} \right) + i \sin \left(\frac{\pi}{12} \right) \right), \\ z_1 &= 2^{1/6} \left(\cos \left(\frac{9\pi}{12} \right) + i \sin \left(\frac{9\pi}{12} \right) \right) = 2^{1/6} \left(\cos \left(\frac{3\pi}{4} \right) + i \sin \left(\frac{3\pi}{4} \right) \right), \\ z_2 &= 2^{1/6} \left(\cos \left(\frac{17\pi}{12} \right) + i \sin \left(\frac{17\pi}{12} \right) \right) = 2^{1/6} \left(\cos \left(-\frac{7\pi}{12} \right) + i \sin \left(-\frac{7\pi}{12} \right) \right). \end{aligned}$$

Selected answers and hints.

- (a) Using the quadratic formula, $z = -1 \pm i\sqrt{3}$.
 (b) Rearranging, we get $(w^2)^2 + 2(w^2) + 4 = 0$, so $w^2 = -1 \pm i\sqrt{3}$. Dealing with each case in turn gives four solutions, namely
 $w = \sqrt{2}e^{\frac{\pi i}{3}}$, $w = \sqrt{2}e^{\frac{2\pi i}{3}}$, $w = \sqrt{2}e^{\frac{-\pi i}{3}}$ and $w = \sqrt{2}e^{\frac{-2\pi i}{3}}$.

- (a) $\cos(3\theta) = 4\cos^3\theta - 3\cos\theta$.
 (b) Use the identity in (a), but replacing θ with $\theta/3$ to get
 $0 = \sin\theta = \sin(3\cdot\theta/3) = -4\sin^3(\theta/3) + 3\sin(\theta/3)$.
 Thus $\sin(\theta/3)\{3 - 4\sin^2(\theta/3)\} = 0$, so $\sin(\theta/3) = 0$ or $\sin(\theta/3) = \pm\sqrt{3}/2$.
 (c) $\sin(5\theta) = 16\sin^5(\theta) - 20\sin^3(\theta) + 5\sin(\theta)$, and a similar method to part (b) will work. One can check that $\sin^2(\pi/5) = (5 - \sqrt{5})/8$.

- (a) Writing $1 = \cos(2k\pi) + i\sin(2k\pi)$, we find that solutions to $z^5 = 1$ are $z = e^{2k\pi i/5}$ for $k = 0, 1, 2, 3, 4$. Here, $\omega = e^{2\pi i/5}$. Multiplying a complex number by ω rotates the number by $2\pi/5$ in the anticlockwise direction.

(b) The solutions are

$$\begin{aligned} z_0 &= 2^{1/5} \left(\cos \left(\frac{\pi}{15} \right) - i \sin \left(\frac{\pi}{15} \right) \right), \\ z_1 &= 2^{1/5} \left(\cos \left(\frac{7\pi}{15} \right) - i \sin \left(\frac{7\pi}{15} \right) \right), \\ z_2 &= 2^{1/5} \left(\cos \left(\frac{13\pi}{15} \right) - i \sin \left(\frac{13\pi}{15} \right) \right), \\ z_3 &= 2^{1/5} \left(\cos \left(\frac{19\pi}{15} \right) - i \sin \left(\frac{19\pi}{15} \right) \right) = 2^{1/5} \left(\cos \left(\frac{11\pi}{15} \right) + i \sin \left(\frac{11\pi}{15} \right) \right), \\ z_4 &= 2^{1/5} \left(\cos \left(\frac{25\pi}{15} \right) - i \sin \left(\frac{25\pi}{15} \right) \right) = 2^{1/5} \left(\cos \left(\frac{5\pi}{15} \right) + i \sin \left(\frac{5\pi}{15} \right) \right). \end{aligned}$$

(c) If ω is as in (a), then $\omega^5 = 1$. Hence $(\omega z)^5 = \omega^5 z^5 = 1 \cdot (1 - i\sqrt{3}) = 1 - i\sqrt{3}$. The fifth roots of $1 - i\sqrt{3}$ will be $z, \omega z, \omega^2 z, \omega^3 z$ and $\omega^4 z$, as each root is obtained from the last by multiplication through a fifth of a full turn, i.e. $2\pi/5$.

- By de Moivre's Theorem, $\omega^n = (\cos(\frac{2\pi}{n}) + i \sin(\frac{2\pi}{n}))^n = \cos(2\pi) + i \sin(2\pi) = 1$. Thus $\omega^n - 1 = 0$. Since

$$z^n - 1 = (z - 1)(z^{n-1} + z^{n-2} + \dots + z + 1)$$

it follows that $(\omega - 1)(1 + \omega + \omega^2 + \dots + \omega^{n-1}) = 0$. But $\omega \neq 1$, so $1 + \omega + \omega^2 + \dots + \omega^{n-1} = 0$. Equating real parts, and using $\omega^k = \cos(2k\pi/n) + i \sin(2k\pi/n)$, we see that $1 + \cos(2\pi/n) + \cos(4\pi/n) + \dots + \cos(2(n-1)\pi/n) = 0$, so $\cos(2\pi/n) + \cos(4\pi/n) + \dots + \cos(2(n-1)\pi/n) = -1$.

For more details, start a thread on the discussion board.