DE MOIVRE’S THEOREM

5 minute review. Recap de Moivre’s Theorem, \( \cos(n\theta) + i \sin(n\theta) = (\cos \theta + i \sin \theta)^n \), and how to solve \( z^n = r(\cos \theta + i \sin \theta) \) for \( z \), perhaps by doing the warm-up below.

Class warm-up. Find the cube roots of \( 1 + i \).

Problems. Choose from the below.

1. (a) Find the complex solutions of \( z^2 + 2z + 4 = 0 \) and draw them on an argand diagram.
   
   (b) Hence find all the solutions to \( w^2 = -2 - \frac{4}{w^2} \).

   
   (a) Use de Moivre’s Theorem to show that \( \sin(3\theta) = -4\sin^3(\theta) + 3\sin(\theta) \).
       What is \( \cos(3\theta) \) in terms of powers of \( \cos \theta \)?
   
   (b) Suppose \( \sin \theta = 0 \). Deduce that \( \sin(\theta/3) = \pm\sqrt{3}/2 \) or 0. Find values of \( \theta \) that give each of these answers.
   
   (c) Find \( \sin(5\theta) \) in terms of powers of \( \sin \theta \) and deduce that
       \[
       \sin^2(\pi/5) = (5 \pm \sqrt{5})/8 \text{ or } 0.
       \]
       Which is it?

3. Roots of unity. For a positive integer \( n \), the \( n \)th-roots of unity are defined to be the solutions to \( z^n = 1 \). There are always \( n \) such solutions.
   
   (a) Find the 5 fifth-roots of unity and plot them on the argand plane. Let \( \omega \) be the solution with the smallest positive argument. What is the effect in the argand plane of multiplying a complex number by \( \omega \)?
   
   (b) Find the 5 fifth-roots of \( 1 - i\sqrt{3} \) and plot them on the argand plane.
   
   (c) Show that if \( \omega \) is as in (a) and \( z \) is a fifth-root of \( 1 - i\sqrt{3} \) then \( \omega z \) is also a fifth-root of \( 1 - i\sqrt{3} \). Can you express the other fifth-roots of \( 1 - i\sqrt{3} \) in terms of \( z \) and \( \omega \)? (Hint: think about multiplication by \( \omega \).)

4. More roots of unity. Let \( n > 1 \) be a positive integer, and let \( \omega = \cos(\frac{2\pi}{n}) + i\sin(\frac{2\pi}{n}) \). By considering the expansion of \( (z - 1)(z^{n-1} + z^{n-2} + \ldots + z + 1) \) show that

\[
1 + \omega + \omega^2 + \ldots + \omega^{n-1} = 0
\]

and hence find a value for \( \cos(\frac{2\pi}{n}) + \cos(\frac{4\pi}{n}) + \ldots + \cos(\frac{2(n-1)\pi}{n}) \).
For the warm-up, \(1 + i = \sqrt{2} \cos(\pi/4) + i \sin(\pi/4)\), so the roots are of the form
\[
z_p = \sqrt[1/3]{2} \left(\cos\left(\frac{\pi}{4} + \frac{2p\pi}{3}\right) + i \sin\left(\frac{\pi}{4} + \frac{2p\pi}{3}\right)\right) \quad \text{for } p = 0, 1, 2.
\]

In other words, we have
\[
z_0 = 2^{1/6} \left(\cos\left(\frac{\pi}{12}\right) + i \sin\left(\frac{\pi}{12}\right)\right),
\]
\[
z_1 = 2^{1/6} \left(\cos\left(\frac{5\pi}{12}\right) + i \sin\left(\frac{5\pi}{12}\right)\right) = 2^{1/6} \left(\cos\left(\frac{3\pi}{12}\right) + i \sin\left(\frac{3\pi}{12}\right)\right),
\]
\[
z_2 = 2^{1/6} \left(\cos\left(\frac{17\pi}{12}\right) + i \sin\left(\frac{17\pi}{12}\right)\right) = 2^{1/6} \left(\cos\left(-\frac{7\pi}{12}\right) + i \sin\left(-\frac{7\pi}{12}\right)\right).
\]

Selected answers and hints.

1. (a) Using the quadratic formula, \(z = -1 \pm i\sqrt{3}\).

   (b) Rearranging, we get \((w^2)^2 + 2(w^2) + 4 = 0\), so \(w^2 = -1 \pm i\sqrt{3}\). Dealing with each case in turn gives four four solutions, namely
   
   \(w = \sqrt{2} e^{\frac{i\pi}{3}}, w = \sqrt{2} e^{\frac{2i\pi}{3}}, w = \sqrt{2} e^{-\frac{i\pi}{3}}\) and \(w = \sqrt{2} e^{-\frac{2i\pi}{3}}\).

2. (a) \(\cos(3\theta) = 4 \cos^3\theta - 3 \cos\theta\).

   (b) Use the identity in (a), but replacing \(\theta\) with \(\theta/3\) to get
   \[
   0 = \sin\theta = \sin(3\theta/3) = -4 \sin^3(\theta/3) + 3 \sin(\theta/3).
   \]
   Thus \(\sin(\theta/3)\{3 - 4 \sin^2(\theta/3)\} = 0\), so \(\sin(\theta/3) = 0\) or \(\sin(\theta/3) = \pm \sqrt{3}/2\).

   (c) \(\sin(5\theta) = 16 \sin^5(\theta) - 20 \sin^3(\theta) + 5 \sin(\theta),\) and a similar method to part (b) will work. One can check that \(\sin^2(\pi/5) = (5 - \sqrt{5})/8\).

3. (a) Writing \(1 = \cos(2k\pi) + i \sin(2k\pi),\) we find that solutions to \(z^5 = 1\) are \(z = e^{2k\pi i/5}\) for \(k = 0, 1, 2, 3, 4\). Here, \(\omega = e^{2\pi i/5}\). Multiplying a complex number by \(\omega\) rotates the number by \(2\pi/5\) in the anticlockwise direction.

   (b) The solutions are
   \[
z_0 = 2^{1/5} \left(\cos\left(\frac{\pi}{15}\right) - i \sin\left(\frac{\pi}{15}\right)\right),
\]
\[
z_1 = 2^{1/5} \left(\cos\left(\frac{7\pi}{15}\right) - i \sin\left(\frac{7\pi}{15}\right)\right),
\]
\[
z_2 = 2^{1/5} \left(\cos\left(\frac{13\pi}{15}\right) - i \sin\left(\frac{13\pi}{15}\right)\right),
\]
\[
z_3 = 2^{1/5} \left(\cos\left(\frac{19\pi}{15}\right) - i \sin\left(\frac{19\pi}{15}\right)\right) = 2^{1/5} \left(\cos\left(\frac{11\pi}{15}\right) + i \sin\left(\frac{11\pi}{15}\right)\right),
\]
\[
z_4 = 2^{1/5} \left(\cos\left(\frac{25\pi}{15}\right) - i \sin\left(\frac{25\pi}{15}\right)\right) = 2^{1/5} \left(\cos\left(\frac{5\pi}{15}\right) + i \sin\left(\frac{5\pi}{15}\right)\right).
\]

   (c) If \(\omega\) is as in (a), then \(\omega^5 = 1\). Hence \((\omega z)^5 = \omega^5 z^5 = 1\). \(1 - i\sqrt{3}\) will be \(z, \omega z, \omega^2 z, \omega^3 z\) and \(\omega^4 z,\) as each root is obtained from the last by multiplication through a fifth of a full turn, i.e. \(2\pi/5\).

4. By de Moivre’s Theorem, \(\omega^n = (\cos(\frac{2n\pi}{n}) + i \sin(\frac{2n\pi}{n})) = \cos(2\pi) + i \sin(2\pi) = 1.\) Thus \(\omega^n - 1 = 0.\) Since
   \[
z^n - 1 = (z - 1)(z^{n-1} + z^{n-2} + \ldots + z + 1)
   \]
   it follows that \((\omega - 1)(1 + \omega + \omega^2 + \ldots + \omega^{n-1}) = 0.\) But \(\omega \neq 1,\) so \(1 + \omega + \omega^2 + \ldots + \omega^{n-1} = 0.\) Equating real parts, and using \(\omega^k = \cos(2k\pi/n) + i \sin(2k\pi/n),\) we see that \(1 + \cos(2\pi/n) + \cos(4\pi/n) + \ldots + \cos(2(n-1)\pi/n) = 0,\) so \(\cos(2\pi/n) + \cos(4\pi/n) + \ldots + \cos(2(n-1)\pi/n) = -1.\)

For more details, start a thread on the discussion board.