DE MOIVRE'S THEOREM

5 minute review. Recap de Moivre's Theorem, $\cos(n\theta) + i\sin(n\theta) = (\cos\theta + i\sin\theta)^n$, and how to solve $z^n = r(\cos\theta + i\sin\theta)$ for z, perhaps by doing the warm-up below.

Class warm-up. Find the cube roots of $1 + i$.

Problems. Choose from the below.

1. (a) Find the complex solutions of $z^2 + 2z + 4 = 0$ and draw them on an Argand diagram.

   (b) Hence find all the solutions to $w^2 = -2 - \dfrac{4}{w^2}$.

2. **More trigonometric identities**.

   (a) Use de Moivre's Theorem to show that $\sin(3\theta) = -4\sin^3(\theta) + 3\sin(\theta)$. What is $\cos(3\theta)$ in terms of powers of $\cos\theta$?

   (b) Suppose $\sin\theta = 0$. Deduce that $\sin(\theta/3) = \pm\sqrt{3}/2$ or 0. Find values of θ that give each of these answers.

   (c) Find $\sin(5\theta)$ in terms of powers of $\sin\theta$ and deduce that

   $$\sin^2(\pi/5) = (5 \pm \sqrt{5})/8 \text{ or } 0.$$

   Which is it?

3. **Roots of unity**. For a positive integer n, the *nth-roots of unity* are defined to be the solutions to $z^n = 1$. There are always n such solutions.

   (a) Find the 5 fifth-roots of unity and plot them on the Argand plane. Let ω be the solution with the smallest positive argument. What is the effect in the Argand plane of multiplying a complex number by ω?

   (b) Find the 5 fifth-roots of $1 - i\sqrt{3}$ and plot them on the Argand plane.

   (c) Show that if ω is as in (a) and z is a fifth-root of $1 - i\sqrt{3}$ then ωz is also a fifth-root of $1 - i\sqrt{3}$. Can you express the other fifth-roots of $1 - i\sqrt{3}$ in terms of z and ω? (Hint: think about multiplication by ω.)

4. **More roots of unity**. Let $n > 1$ be a positive integer, and let $\omega = \cos(\frac{2\pi}{n}) + i\sin(\frac{2\pi}{n})$. By considering the expansion of $(z-1)(z^{n-1} + z^{n-2} + \ldots + z + 1)$ show that

   $$1 + \omega + \omega^2 + \ldots + \omega^{n-1} = 0$$

   and hence find a value for $\cos(\frac{2\pi}{n}) + \cos(\frac{4\pi}{n}) + \ldots + \cos(\frac{2(n-1)\pi}{n})$.

For the warm-up, $1+i=\sqrt{2}(\cos(\pi/4)+i\sin(\pi/4))$, so the roots are of the form

$$z_p = \sqrt{2}^{1/3}\left(\cos\left(\frac{\pi/4+2p\pi}{3}\right)+i\sin\left(\frac{\pi/4+2p\pi}{3}\right)\right) \quad \text{for } p=0,1,2.$$

In other words, we have

$$\begin{aligned}
z_0 &= 2^{1/6}\left(\cos\left(\tfrac{\pi}{12}\right)+i\sin\left(\tfrac{\pi}{12}\right)\right),\\
z_1 &= 2^{1/6}\left(\cos\left(\tfrac{9\pi}{12}\right)+i\sin\left(\tfrac{9\pi}{12}\right)\right) = 2^{1/6}\left(\cos\left(\tfrac{3\pi}{4}\right)+i\sin\left(\tfrac{3\pi}{4}\right)\right),\\
z_2 &= 2^{1/6}\left(\cos\left(\tfrac{17\pi}{12}\right)+i\sin\left(\tfrac{17\pi}{12}\right)\right) = 2^{1/6}\left(\cos\left(-\tfrac{7\pi}{12}\right)+i\sin\left(-\tfrac{7\pi}{12}\right)\right).
\end{aligned}$$

Selected answers and hints.

1. (a) Using the quadratic formula, $z=-1\pm i\sqrt{3}$.

   (b) Rearranging, we get $(w^2)^2+2(w^2)+4=0$, so $w^2=-1\pm i\sqrt{3}$. Dealing with each case in turn gives four solutions, namely

   $$w=\sqrt{2}e^{\frac{\pi i}{3}},\ w=\sqrt{2}e^{\frac{2\pi i}{3}},\ w=\sqrt{2}e^{\frac{-\pi i}{3}} \text{ and } w=\sqrt{2}e^{\frac{-2\pi i}{3}}.$$

2. (a) $\cos(3\theta)=4\cos^3\theta-3\cos\theta$.

   (b) Use the identity in (a), but replacing θ with $\theta/3$ to get

   $$0=\sin\theta=\sin(3.\theta/3)=-4\sin^3(\theta/3)+3\sin(\theta/3).$$

   Thus $\sin(\theta/3)\{3-4\sin^2(\theta/3)\}=0$, so $\sin(\theta/3)=0$ or $\sin(\theta/3)=\pm\sqrt{3}/2$.

   (c) $\sin(5\theta)=16\sin^5(\theta)-20\sin^3(\theta)+5\sin(\theta)$, and a similar method to part (b) will work. One can check that $\sin^2(\pi/5)=(5-\sqrt{5})/8$.

3. (a) Writing $1=\cos(2k\pi)+i\sin(2k\pi)$, we find that solutions to $z^5=1$ are $z=e^{2k\pi i/5}$ for $k=0,1,2,3,4$. Here, $\omega=e^{2\pi i/5}$. Multiplying a complex number by ω rotates the number by $2\pi/5$ in the anticlockwise direction.

   (b) The solutions are

   $$\begin{aligned}
   z_0 &= 2^{1/5}\left(\cos\left(\tfrac{\pi}{15}\right)-i\sin\left(\tfrac{\pi}{15}\right)\right),\\
   z_1 &= 2^{1/5}\left(\cos\left(\tfrac{7\pi}{15}\right)-i\sin\left(\tfrac{7\pi}{15}\right)\right),\\
   z_2 &= 2^{1/5}\left(\cos\left(\tfrac{13\pi}{15}\right)-i\sin\left(\tfrac{13\pi}{15}\right)\right),\\
   z_3 &= 2^{1/5}\left(\cos\left(\tfrac{19\pi}{15}\right)-i\sin\left(\tfrac{19\pi}{15}\right)\right) = 2^{1/5}\left(\cos\left(\tfrac{11\pi}{15}\right)+i\sin\left(\tfrac{11\pi}{15}\right)\right),\\
   z_4 &= 2^{1/5}\left(\cos\left(\tfrac{25\pi}{15}\right)-i\sin\left(\tfrac{25\pi}{15}\right)\right) = 2^{1/5}\left(\cos\left(\tfrac{5\pi}{15}\right)+i\sin\left(\tfrac{5\pi}{15}\right)\right).
   \end{aligned}$$

   (c) If ω is as in (a), then $\omega^5=1$. Hence $(\omega z)^5=\omega^5z^5=1.(1-i\sqrt{3})=1-i\sqrt{3}$. The fifth roots of $1-i\sqrt{3}$ will be z, ωz, $\omega^2 z$, $\omega^3 z$ and $\omega^4 z$, as each root is obtained from the last by multiplication through a fifth of a full turn, i.e. $2\pi/5$.

4. By de Moivre's Theorem, $\omega^n=(\cos(\frac{2\pi}{n})+i\sin(\frac{2\pi}{n}))^n=\cos(2\pi)+i\sin(2\pi)=1$. Thus $\omega^n-1=0$. Since

   $$z^n-1=(z-1)(z^{n-1}+z^{n-2}+\ldots+z+1)$$

   it follows that $(\omega-1)(1+\omega+\omega^2+\ldots+\omega^{n-1})=0$. But $\omega\neq 1$, so $1+\omega+\omega^2+\ldots+\omega^{n-1}=0$. Equating real parts, and using $\omega^k=\cos(2k\pi/n)+i\sin(2k\pi/n)$, we see that $1+\cos(2\pi/n)+\cos(4\pi/n)+\ldots+\cos(2(n-1)\pi/n)=0$, so $\cos(2\pi/n)+\cos(4\pi/n)+\ldots+\cos(2(n-1)\pi/n)=-1$.

For more details, start a thread on the discussion board.