EULER’S RELATION

5 minute review. Review Euler’s relation, $e^{i\theta} = \cos \theta + i \sin \theta$, commenting briefly on how it follows from the Maclaurin series of exp, sin and cos. Discuss how this means that any complex number can be written in exponential form, $re^{i\theta}$. Also cover the exponential identities for sin and cos, namely

$$\sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \quad \text{and} \quad \cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2}.$$

Class warm-up. With the aid of a diagram, find the conjugate of $re^{i\theta}$. What shape in the complex plane does $e^{(1+i)\theta}$ trace out as $\theta$ varies? What about $e^{(1+i)\theta}$?

Problems. Choose from the below.

1. Relating functions. What are $\sin(i\theta)$ and $\cos(i\theta)$? What about $\tan(i\theta)$? And $\tanh(i\theta)$?

2. Trigonometric identities. Recall that we can use the exponential form of cos together with the binomial theorem to show that $\cos^3(\theta) = \frac{1}{4} \cos(3\theta) + \frac{3}{4} \cos(\theta)$. Use the same method to fill in the question marks in the identity

$$\cos^5(\theta) = ? \cos(5\theta) + ? \cos(3\theta) + ? \cos(\theta)$$

and find a general formula for $\cos^n(\theta)$ for any odd positive integer $n$.

3. Exponential form and negative numbers.
   (a) Let $z = 3e^{3\pi i}$. Plot $z$ on the argand diagram.
   (b) What is $e^{i\pi}$?
   (c) Now let $z = -3e^{3\pi i}$. Plot $z$ on the argand diagram. What is $|z|$? What is arg($z$)? Write $z$ in polar and exponential form.

4. More trigonometric identities. Prove the addition formulae for $\sin(A+B)$ and $\cos(A+B)$ by using the exponential forms of sin and cos.

5. More on exponential form. Let $z = e^{3\ln \theta + (\theta - \pi)i}$ for $-\pi < \theta < \pi$.
   (a) What is $|z|$? What is arg($z$)? (Hint: you will get different answers depending on the sign of $\theta$.)
   (b) Draw the shape in the complex plane that $z$ traces out as $\theta$ varies.
For the warm-up, \( e^{(1+i)\theta} = e^{\theta+i\theta} = e^\theta e^{i\theta} \). This traces out a spiral. Conjugating a number reflects it in the real axis, so \( e^{(1+i)\overline{\theta}} = e^{\overline{\theta}} e^{i\theta} = e^\theta e^{-i\theta} \) will be the reflection of the spiral in the real axis.

Selected answers and hints.

1. From the exponential forms, \( \sin(i\theta) = i\sinh \theta \) and \( \cos(i\theta) = \cosh \theta \). Hence \( \tan(i\theta) = \sin(i\theta)/\cos(i\theta) = i\tanh \theta \), and so
   \[
   \tanh(i\theta) = \frac{1}{i} \tan(i^2 \theta) = \frac{i}{i^2} \tan(-\theta) = -i \tan(-\theta) = i \tan(\theta).
   \]

2. \( \cos^5(\theta) = \frac{1}{16} \cos(5\theta) + \frac{5}{16} \cos(3\theta) + \frac{5}{8} \cos(\theta). \) Using the same method, it follows that
   \[
   \cos^n(\theta) = \frac{1}{2^{n-1}} \left( \cos(n\theta) + \left(\frac{n}{1}\right) \cos((n-2)\theta) + \left(\frac{n}{2}\right) \cos((n-4)\theta) + \cdots + \left(\frac{n}{n-1}\right) \cos(\theta) \right).
   \]

3. (a) \( z = 3e^{3\pi i} \) has modulus 3 and argument \( \frac{3\pi}{4} \).
   (b) \( z = -3e^{3\pi i} \) sits diametrically opposite \( 3e^{3\pi i} \) in the argand plane. Thus it has modulus 3 and argument \( -\frac{\pi}{4} \). That is, \( z = 3(\cos(-\frac{\pi}{4}) + i \sin(-\frac{\pi}{4})) = 3e^{-\frac{\pi}{4}i} \).
   (c) Since \( e^{i\pi} = -1 \), it follows that
   \[
   z = -3e^{\frac{3\pi}{4}i} = e^{i\pi} 3e^{\frac{3\pi}{4}i} = 3e^{\frac{\pi}{4} + \pi i} = 3e^{\frac{7\pi}{4}i} = 3e^{-\frac{\pi}{4}i}.
   \]

5. (a) \( z = e^{3\ln \theta + (\theta - \pi)i} = e^{3\ln \theta} e^{(\theta - \pi)i} = \theta^3 e^{(\theta - \pi)i}. \)
   - When \( \theta = 0 \), \( z = 0 \) (which has modulus 0 and undefined argument).
   - When \( 0 < \theta < \pi \), \( |z| = \theta^3 \) and \( \arg(z) = \theta - \pi. \)
   - When \( -\pi < \theta < 0 \), \( |z| = -\theta^3 \) (since \( \theta \) is negative) and \( z = -(-\theta^3)e^{(\theta - \pi)i} = e^{i\pi}(-\theta^3)e^{(\theta - \pi)i} = (-\theta^3)e^{\theta i}, \) so \( \arg(z) = \theta. \)

Summarising, the exponential form for \( z \) is
   \[
   z = \begin{cases} 
   (-\theta^3)e^{i\theta} & \text{for } -\pi < \theta < 0 \\
   0 & \text{for } \theta = 0 \\
   \theta^3 e^{(\theta - \pi)i} & \text{for } 0 < \theta < \pi 
   \end{cases}
   \]

(b) The diagram is as below.

For more details, start a thread on the discussion board.