EULER'S RELATION

5 minute review. Review Euler's relation, $e^{i\theta} = \cos\theta + i\sin\theta$, commenting briefly on how it follows from the Maclaurin series of exp, sin and cos. Discuss how this means that any complex number can be written in *exponential form*, $re^{i\theta}$. Also cover the exponential identities for sin and cos, namely

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \quad \text{and} \quad \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}.$$

Class warm-up. With the aid of a diagram, find the conjugate of $re^{i\theta}$. What shape in the complex plane does $e^{(1+i)\theta}$ trace out as θ varies? What about $\overline{e^{(1+i)\theta}}$?

Problems. Choose from the below.

1. **Relating functions**. What are $\sin(i\theta)$ and $\cos(i\theta)$? What about $\tan(i\theta)$? And $\tanh(i\theta)$?
2. **Trigonometric identities**. Recall that we can use the exponential form of cos together with the binomial theorem to show that $\cos^3(\theta) = \frac{1}{4}\cos(3\theta) + \frac{3}{4}\cos(\theta)$. Use the same method to fill in the question marks in the identity
$$\cos^5(\theta) = ?\cos(5\theta) + ?\cos(3\theta) + ?\cos(\theta)$$
and find a general formula for $\cos^n(\theta)$ for any odd positive integer n.
3. **Exponential form and negative numbers**.
   (a) Let $z = 3e^{\frac{3\pi}{4}i}$. Plot z on the argand diagram.
   (b) What is $e^{i\pi}$?
   (c) Now let $z = -3e^{\frac{3\pi}{4}i}$. Plot z on the argand diagram. What is $|z|$? What is $\arg(z)$? Write z in polar and exponential form.
4. **More trigonometric identities**. Prove the addition formulae for $\sin(A+B)$ and $\cos(A+B)$ by using the exponential forms of sin and cos.
5. **More on exponential form**. Let $z = e^{3\ln\theta + (\theta-\pi)i}$ for $-\pi < \theta < \pi$.
   (a) What is $|z|$? What is $\arg(z)$? (Hint: you will get different answers depending on the sign of θ.)
   (b) Draw the shape in the complex plane that z traces out as θ varies.

For the warm-up, $e^{(1+i)\theta} = e^{\theta+i\theta} = e^{\theta}e^{i\theta}$. This traces out a spiral. Conjugating a number reflects it in the real axis, so $\overline{e^{(1+i)\theta}} = \overline{e^{\theta}e^{i\theta}} = e^{\theta}e^{-i\theta}$ will be the reflection of the spiral in the real axis.

Selected answers and hints.

1. From the exponential forms, $\sin(i\theta) = i\sinh\theta$ and $\cos(i\theta) = \cosh\theta$. Hence $\tan(i\theta) = \sin(i\theta)/\cos(i\theta) = i\tanh\theta$, and so
$$\tanh(i\theta) = \frac{1}{i}\tan(i^2\theta) = \frac{i}{i^2}\tan(-\theta) = -i\tan(-\theta) = i\tan(\theta).$$

2. $\cos^5(\theta) = \frac{1}{16}\cos(5\theta) + \frac{5}{16}\cos(3\theta) + \frac{5}{8}\cos(\theta)$. Using the same method, it follows that
$$\cos^n(\theta) = \frac{1}{2^{n-1}}\left(\cos(n\theta) + \binom{n}{1}\cos((n-2)\theta) + \binom{n}{2}\cos((n-4)\theta) + \cdots + \binom{n}{\frac{n-1}{2}}\cos(\theta)\right).$$

3. (a) $z = 3e^{\frac{3\pi}{4}i}$ has modulus 3 and argument $\frac{3\pi}{4}$.

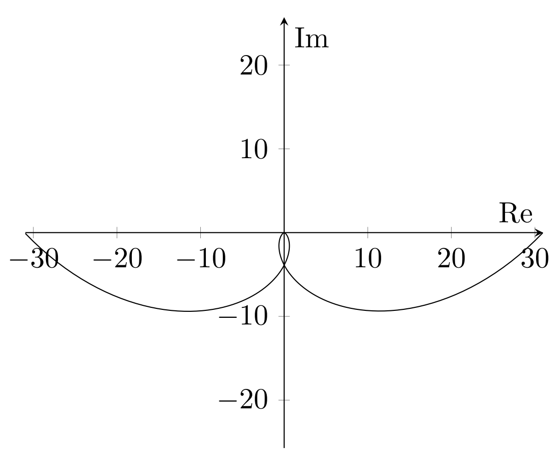
   (b) $z = -3e^{\frac{3\pi}{4}i}$ sits diametrically opposite $3e^{\frac{3\pi}{4}i}$ in the argand plane. Thus it has modulus 3 and argument $-\frac{\pi}{4}$. That is, $z = 3(\cos(-\frac{\pi}{4}) + i\sin(-\frac{\pi}{4})) = 3e^{-\frac{\pi}{4}i}$.

   (c) Since $e^{i\pi} = -1$, it follows that
$$z = -3e^{\frac{3\pi}{4}i} = e^{i\pi}.3e^{\frac{3\pi}{4}i} = 3e^{(\frac{3\pi}{4}+\pi)i} = 3e^{\frac{7\pi}{4}i} = 3e^{-\frac{\pi}{4}i}.$$

5. (a) $z = e^{3\ln\theta + (\theta-\pi)i} = e^{3\ln\theta}e^{(\theta-\pi)i} = \theta^3 e^{(\theta-\pi)i}$.
   - When $\theta = 0$, $z = 0$ (which has modulus 0 and undefined argument).
   - When $0 < \theta < \pi$, $|z| = \theta^3$ and $\arg(z) = \theta - \pi$.
   - When $-\pi < \theta < 0$, $|z| = -\theta^3$ (since θ is negeative) and $z = -(-\theta^3)e^{(\theta-\pi)i} = e^{i\pi}.(-\theta^3)e^{(\theta-\pi)i} = (-\theta^3)e^{\theta i}$, so $\arg z = \theta$.

   Summarising, the exponential form for z is
$$z = \begin{cases} (-\theta^3)e^{i\theta} & \text{for } -\pi < \theta < 0 \\ 0 & \text{for } \theta = 0 \\ \theta^3 e^{(\theta-\pi)i} & \text{for } 0 < \theta < \pi \end{cases}$$

   (b) The diagram is as below.



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