

EULER'S RELATION

5 minute review. Review Euler's relation, $e^{i\theta} = \cos \theta + i \sin \theta$, commenting briefly on how it follows from the Maclaurin series of \exp , \sin and \cos . Discuss how this means that any complex number can be written in *exponential form*, $re^{i\theta}$. Also cover the exponential identities for \sin and \cos , namely

$$\sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \quad \text{and} \quad \cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2}.$$

Class warm-up. With the aid of a diagram, find the conjugate of $re^{i\theta}$. What shape in the complex plane does $e^{(1+i)\theta}$ trace out as θ varies? What about $\overline{e^{(1+i)\theta}}$?

Problems. Choose from the below.

1. **Relating functions.** What are $\sin(i\theta)$ and $\cos(i\theta)$? What about $\tan(i\theta)$? And $\tanh(i\theta)$?

2. **Trigonometric identities.** Recall that we can use the exponential form of \cos together with the binomial theorem to show that $\cos^3(\theta) = \frac{1}{4}\cos(3\theta) + \frac{3}{4}\cos(\theta)$. Use the same method to fill in the question marks in the identity

$$\cos^5(\theta) = ? \cos(5\theta) + ? \cos(3\theta) + ? \cos(\theta)$$

and find a general formula for $\cos^n(\theta)$ for any odd positive integer n .

3. **Exponential form and negative numbers.**

(a) Let $z = 3e^{\frac{3\pi}{4}i}$. Plot z on the argand diagram.

(b) What is $e^{i\pi}$?

(c) Now let $z = -3e^{\frac{3\pi}{4}i}$. Plot z on the argand diagram. What is $|z|$? What is $\arg(z)$? Write z in polar and exponential form.

4. **More trigonometric identities.** Prove the addition formulae for $\sin(A+B)$ and $\cos(A+B)$ by using the exponential forms of \sin and \cos .

5. **More on exponential form.** Let $z = e^{3\ln \theta + (\theta - \pi)i}$ for $-\pi < \theta < \pi$.

(a) What is $|z|$? What is $\arg(z)$? (Hint: you will get different answers depending on the sign of θ .)

(b) Draw the shape in the complex plane that z traces out as θ varies.

For the warm-up, $e^{(1+i)\theta} = e^{\theta+i\theta} = e^{\theta}e^{i\theta}$. This traces out a spiral. Conjugating a number reflects it in the real axis, so $e^{(1+i)\theta} = e^{\theta}e^{i\theta} = e^{\theta}e^{-i\theta}$ will be the reflection of the spiral in the real axis.

Selected answers and hints.

1. From the exponential forms, $\sin(i\theta) = i \sinh \theta$ and $\cos(i\theta) = \cosh \theta$. Hence $\tan(i\theta) = \sin(i\theta)/\cos(i\theta) = i \tanh \theta$, and so

$$\tanh(i\theta) = \frac{1}{i} \tan(i^2\theta) = \frac{i}{i^2} \tan(-\theta) = -i \tan(-\theta) = i \tan(\theta).$$

2. $\cos^5(\theta) = \frac{1}{16} \cos(5\theta) + \frac{5}{16} \cos(3\theta) + \frac{5}{8} \cos(\theta)$. Using the same method, it follows that

$$\cos^n(\theta) = \frac{1}{2^{n-1}} \left(\cos(n\theta) + \binom{n}{1} \cos((n-2)\theta) + \binom{n}{2} \cos((n-4)\theta) + \cdots + \binom{n}{\frac{n-1}{2}} \cos(\theta) \right).$$

3. (a) $z = 3e^{\frac{3\pi}{4}i}$ has modulus 3 and argument $\frac{3\pi}{4}$.
 (b) $z = -3e^{\frac{3\pi}{4}i}$ sits diametrically opposite $3e^{\frac{3\pi}{4}i}$ in the argand plane. Thus it has modulus 3 and argument $-\frac{\pi}{4}$. That is, $z = 3(\cos(-\frac{\pi}{4}) + i \sin(-\frac{\pi}{4})) = 3e^{-\frac{\pi}{4}i}$.
 (c) Since $e^{i\pi} = -1$, it follows that

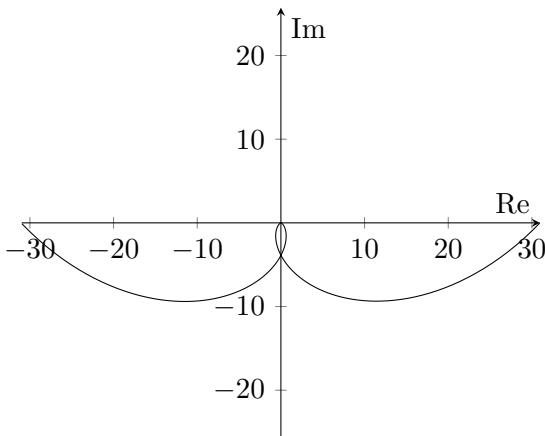
$$z = -3e^{\frac{3\pi}{4}i} = e^{i\pi} \cdot 3e^{\frac{3\pi}{4}i} = 3e^{(\frac{3\pi}{4}+\pi)i} = 3e^{\frac{7\pi}{4}i} = 3e^{-\frac{\pi}{4}i}.$$

5. (a) $z = e^{3 \ln \theta + (\theta - \pi)i} = e^{3 \ln \theta} e^{(\theta - \pi)i} = \theta^3 e^{(\theta - \pi)i}$.
- When $\theta = 0$, $z = 0$ (which has modulus 0 and undefined argument).
 - When $0 < \theta < \pi$, $|z| = \theta^3$ and $\arg(z) = \theta - \pi$.
 - When $-\pi < \theta < 0$, $|z| = -\theta^3$ (since θ is negative) and $z = -(-\theta^3)e^{(\theta - \pi)i} = e^{i\pi} \cdot (-\theta^3)e^{(\theta - \pi)i} = (-\theta^3)e^{\theta i}$, so $\arg z = \theta$.

Summarising, the exponential form for z is

$$z = \begin{cases} (-\theta^3)e^{i\theta} & \text{for } -\pi < \theta < 0 \\ 0 & \text{for } \theta = 0 \\ \theta^3 e^{(\theta - \pi)i} & \text{for } 0 < \theta < \pi \end{cases}$$

- (b) The diagram is as below.



For more details, start a thread on the discussion board.