THE ARGAND DIAGRAM

5 minute review. Review the Argand diagram (aka Argand plane, aka complex plane), the modulus and argument and the polar form of a complex number as $z = r(\cos\theta + i\sin\theta)$. Also cover the geometric effect of addition and multiplication.

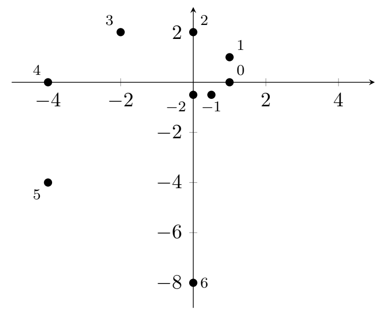
Class warm-up. With the help of students, calculate i^n for $n = 1, \ldots, 6$, then plot them on the argand plane. Do the same for $n = -1, -2$, etc. Next plot $(1+i)^n$ for $n = 1, \ldots, 6$. Then plot $(1+i)^0$ and ask students to guess where $(1+i)^{-1}$ and $(1+i)^{-2}$ sit. Confirm with calculations.

Problems. Choose from the below.

1. **Plotting complex numbers**. Calculate the modulus and argument of the following, plotting them on the argand diagram.
$$\text{(a) } \sqrt{3}+i, \text{ (b) } -\sqrt{2}+\sqrt{6}i, \text{ (c) } \frac{1}{2+3i}.$$
2. **Loci**.
   (a) What does the set $\{z \in \mathbb{C} : |z| = 3\}$ look like in the Argand plane (that is, those points in the complex plane for which $|z| = 3$)?
   (b) For z and w complex numbers, plot z, w, $-w$ and $z - w$ on the Argand plane. What is the geometric meaning of $|z - w|$? Which complex numbers z satisfy $|z - i| = |z - 1|$? Draw them on the complex plane.
3. **Sums of roots**.
   (a) Which complex number has modulus 1 and argument $\frac{2\pi}{5}$? What is z^5? Plot z, z^2, z^3, z^4 and z^5 on the Argand diagram. Without calculating, what is $z^4 + z^3 + z^2 + z$? (Hint: $z^5 - 1 = 0$, but $z \neq 1$. Factorise!)
   (b) If $|z| = 2$ and $\arg z = \frac{\pi}{3}$, what is $z^5 + 2z^4 + 4z^3 + 8z^2 + 16z$?
4. **Machin's Formula***.
   (a) Use the Binomial Theorem to write $(5+i)^4$ in the form $a+bi$. Now show that
$$(5+i)^4(239-i) = 114244(1+i).$$
   Take arguments of both sides to show that
$$4\arctan(1/5) - \arctan(1/239) = \pi/4. \qquad (*)$$
   (b) The Maclaurin expansion for $\arctan x$, valid for $|x| \leq 1$, is
$$\arctan x = x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 - \ldots\ .$$
   Use these three terms to calculate (to 8 d.p.) an approximation to the left hand side of $(*)$. Does it look like π after multiplying by 4?

For the warm-up, here is a diagram:

Powers of $1+i$



Selected answers and hints.

1. (a) $\sqrt{3}+i$ has modulus 2 and argument $\frac{\pi}{6}$.

   (b) $-\sqrt{2}+\sqrt{6}i$ has modulus $2\sqrt{2}$ and argument $\frac{2\pi}{3}$.

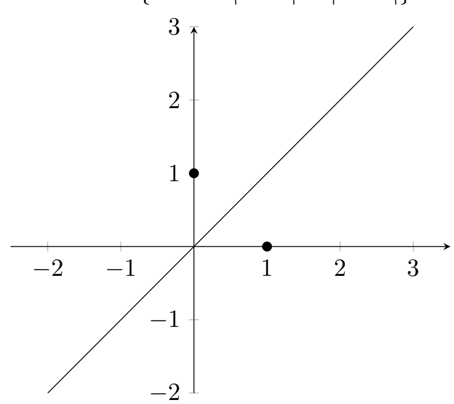
   (c) $\frac{1}{2+3i} = \frac{2-3i}{2^2+3^2} = \frac{2}{13} - \frac{3}{13}i$, which has modulus $\frac{1}{\sqrt{13}}$ and argument $-\tan^{-1}(\frac{3}{2}) \approx -0.98$ (2 s.f.).

2. (a) $\{z \in \mathbb{C} : |z| = 3\}$ is a circle, centred on the origin, radius 3.

   (b) $|z-w|$ gives the distance between z and w.

   $\{z \in \mathbb{C} : |z-i| = |z-1|\}$ is the set of all points whose distance from i is the same as their distance from 1 in the Argand diagram. In other words, it is the perpendicular bisector of those points, namely the diagonal $\mathrm{Re}(z) = \mathrm{Im}(z)$ (i.e. $y = x$).

   The set $\{z \in \mathbb{C} : |z-i| = |z-1|\}$

3. (a) Here, $z = \cos(\frac{2\pi}{5}) + i\sin(\frac{2\pi}{5})$, which is at a distance 1 from the origin, and a fifth of a full turn anticlockwise from the real axis. It follows that z^5 will also have modulus 1 and argument $5 \times \frac{2\pi}{5} = 2\pi$; that is, $z^5 = 1$.

   For the second part, since $z^5 = 1$, it follows that $z^5 - 1 = 0$, which factorises to give $(z-1)(z^4+z^3+z^2+z+1) = 0$. Since $z \neq 1$, that means that $z^4+z^3+z^2+z = -1$.

   (b) Here, z^6 will have modulus $2^6 = 64$ and argument $6 \times \frac{\pi}{3} = 2\pi$, so $z^6 = 64$. Hence, $z^6 - 64 = 0$, which factorises as $(z-2)(z^5+2z^4+4z^3+8z^2+16z+32) = 0$. It follows that $z^5+2z^4+4z^3+8z^2+16z = -32$.

For more details, start a thread on the discussion board.