COMPLEX NUMBERS

5 minute review. Remind students what a complex number is, how to add and multiply them, what the complex conjugate is, and how to do division. They will also need to know that a *root* of a polynomial $f(x)$ is a solution to $f(x) = 0$, and that a is a root of $f(x)$ if and only if $(x - a)$ is a factor of $f(x)$.

Class warm-up. With input from the class, find the roots of the polynomials

$$x^2 - 4, \quad 2x^2 - 10x + 12, \quad x^2 + 4,$$
$$x^2 - 5x + 7, \quad x^4 - 1, \quad x^3 - 7x^2 + 17x - 14.$$

Problems. Choose from the below.

1. **Complex arithmetic**.
   (a) For $z_1 = 3 + 4i$, $z_2 = -1 + i$ and $z_3 = \frac{22}{7} + 7i$, calculate
   $$\overline{z_1 + z_2}, \quad \overline{z_1} + \overline{z_2}, \quad z_1 z_2, \quad z_3/z_2, \quad z_1 - \overline{z_2}, \quad (z_1 + \overline{z_1})/(z_2 z_3).$$
   (b) Find all the complex solutions of each of the following equations. (Hint: write $z = a + ib$ and equate real and imaginary parts.)
   $$z^2 = 3 + 4i, \quad z^2 = \overline{z}.$$
2. **Roots of real polynomials**. Let $f(x) = ax^2 + bx + c$, where $a, b, c \in \mathbb{R}$. The discriminant of $f(x)$ is $\Delta = b^2 - 4ac$.
   (a) What does the sign of Δ tell you about the roots of $f(x)$?
   (b) Suppose that $\Delta < 0$. Find the roots of $f(x)$, and show that they are conjugates.
   (c) Suppose that $z = \alpha + i\beta$ is a root of $f(x)$. Show that $\overline{z}$ is also a root.
   (d) Let $f(x) = a_n x^n + \ldots + a_1 x + a_0$ be any real polynomial of degree n. Show that the same holds here, namely that if $f(z) = 0$ then $f(\overline{z}) = 0$. (Hint: take complex conjugates of both sides.)

   Note that this means that complex roots of any real polynomial must come in conjugate pairs.
3. **Factorizing polynomials***. Let $f(x) = a_n x^n + \ldots + a_1 x + a_0$ be any polynomial with real coefficients of degree n.
   (a) Suppose that $z = 1 + i$ is a root of $f(x)$. Use 2(d) to find a quadratic polynomial which is a factor of $f(x)$.
   (b) Suppose that $f(x)$ has degree 5, and that $z = 1 + i$, $z = 2 - i$ and $z = 2$ are all roots of $f(x)$. What are the other roots? If $f(1) = -4$, what is $f(x)$?
   (c) The fundamental theorem of algebra says that every polynomial has at least one complex root. Deduce that every real polynomial can be factorized as a product of real linear factors and real quadratic factors; that is,
   $$f(x) = A(x - \alpha_1) \ldots (x - \alpha_n)(x^2 + b_1 x + c_1) \ldots (x^2 + b_m x + c_m)$$
   with $A, \alpha_1, \ldots, \alpha_n, b_1, \ldots, b_m, c_1, \ldots, c_m \in \mathbb{R}$.

For the warm-up,

- $x^2 - 4 = (x+2)(x-2)$,
- $2x^2 - 10x + 12 = 2(x-2)(x-3)$,
- $x^2 + 4 = (x+2i)(x-2i)$,
- $x^2 - 5x + 7 = (x - (\frac{5}{2} + \frac{\sqrt{3}}{2}i))(x - (\frac{5}{2} - \frac{\sqrt{3}}{2}i))$,
- $x^4 - 1 = (x+1)(x-1)(x+i)(x-i)$,
- $x^3 - 7x^2 + 17x - 14 = (x-2)(x - (\frac{5}{2} + \frac{\sqrt{3}}{2}i))(x - (\frac{5}{2} - \frac{\sqrt{3}}{2}i))$.

Selected answers and hints.

1. (a) $z_1 z_2 = -7 - i$, $z_3/z_2 = \frac{27}{14} - \frac{71}{14}i$, $z_1 - \overline{z_2} = 4 + 5i$, $(z_1 + \overline{z_1})/(z_2 z_3) = -\frac{1491}{2885} + \frac{567}{2885}i$.

   (b) Let $z = a + ib$. Then $z^2 = 3 + 4i \iff a^2 - b^2 + 2abi = 3 + 4i \iff a^2 - b^2 = 3$ and $2ab = 4 \iff z = 2 + i$ or $z = -2 - i$.

   Similarly $z^2 = \overline{z} \iff a^2 - b^2 + 2abi = a - ib \iff a^2 - b^2 = a$ and $2ab = -b \iff z = 0, 1, -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$.

2. (a) If $\Delta > 0$ there are two distinct real roots; if $\Delta = 0$ then there is a repeated real root; if $\Delta < 0$ then then are two complex (non-real) roots.

   (b) The roots are $x = -\frac{b}{2a} \pm \frac{\sqrt{4ac - b^2}}{2a}i$, which are conjugates of each other.

   (d) Suppose that $f(z) = 0$, that is $a_n z^n + \cdots + a_1 z + a_0 = 0$. Then $\overline{a_n z^n + \cdots + a_1 z + a_0} = \overline{0}$. But $\overline{0} = 0$ so, by the rules of conjugates,

   $$\overline{a_n}.\overline{z}^n + \cdots + \overline{a_1}.\overline{z} + \overline{a_0} = 0$$

   so $a_n \overline{z}^n + \cdots + a_1 \overline{z} + a_0$ since $a_n, \ldots, a_0$ are just real numbers. Thus $f(\overline{z}) = 0$, as claimed.

3. (a) $(x - (1+i))(x - (1-i)) = x^2 - 2x + 2$ is a factor.

   (b) From (a) we know that $x^2 - 2x + 2$ is a factor. The same process shows that $(x - (2-i))(x - (2+i)) = x^2 - 4x + 5$ is also factor, as must be $(x-2)$. It follows that $f(x) = A(x-2)(x^2 - 2x + 2)(x^2 - 4x + 5)$ for some real number A. Putting $x = 1$, $f(x) = -4$ gives $A = 2$, so $f(x) = 2(x-2)(x^2 - 2x + 2)(x^2 - 4x + 5)$.

   (c) Think about the process in (b). Each time you have root, if it's real you know a linear factor, if it's complex you know a quadratic one. Factorising this out, you'll be left with a polynomial of degree 1 or 2 less than before. Carry on until you've finished!

For more details, start a thread on the discussion board.