PARTIAL SECOND DERIVATIVES

Announcement. Remind students that there is a full-class lecture in Week 8 (MAS140 W1 DIA-LT1, MAS151 W1 DIA-LT3, MAS152 M1 DIA-LT4, MAS156(Aero) Tu1 DIA-LT4, MAS156(Elec) & MAS161 M5 DIA-LT4). All should attend.

5 minute review. Remind students how to calculate the second derivatives of $f(x, y)$, the notation f_{xx}, f_{yy} etc, and that $f_{xy} = f_{yx}$ (a least in every case they'll come across). Remind them how to find stationary points ($f_x = f_y = 0$), and that to test them look at $\Delta = f_{xx}f_{yy} - (f_{xy})^2$, with

- $\Delta > 0$ and $f_{xx} > 0$ giving a local minimum;
- $\Delta > 0$ and $f_{xx} < 0$ giving a local maximum;
- $\Delta < 0$ giving a saddle;
- $\Delta = 0$ being inconclusive.

Class warm-up. Find the stationary points of $f(x, y) = 2x^3 + 6xy^2 - 3y^3 - 150x$. Emphasise how good presentational logic is needed to make sure all four stationary points are found. Perhaps leave students to classify the points.

Problems. Choose from the below.

1. **Stationary points**. Find and classify the stationary points of
   (a) $f(x, y) = 3x^2 + xy + 2y$;
   (b) $f(x, y) = e^{-(x^2+y^2)}$;
   (c) $f(x, y) = \cos(xy)$. (Hint: try to think about what the surface $z = f(x, y)$ looks like, investigating well-chosen contours.)
2. **Harmonic functions**. Let $f(x, y) = e^{-y} \cos x$.
   (a) Show that $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$. (Functions satisfying this equation are called *harmonic* functions.)
   (b) Does this function have any stationary points?
   (c) What do slices through the surface $z = f(x, y)$ parallel to the x-axis look like (e.g. where $y = 0$, $y = 1$ etc)? Can you describe the surface overall?
3. **Taylor series**. Just as in the single variable case, it is possible to approximate well-behaved functions of two variables with a *Taylor series* at $(0, 0)$:
$$\begin{aligned} f(x, y) = f(0, 0) &+ xf_x(0, 0) + yf_y(0, 0) \\ &+ \frac{1}{2!}\left(x^2 f_{xx}(0, 0) + 2xyf_{xy}(0, 0) + y^2 f_{yy}(0, 0)\right) + \ldots \end{aligned}$$
   for small x and y.
   (a) Find the first few terms in the Taylor series of $f(x, y) = e^x \ln(1 + y)$. Try some points (x, y) near $(0, 0)$ to see how well your approximation works.
   (b) Can you guess how the Taylor series for $f(x, y)$ continues past the '$\ldots$'?

For the warm-up, there are four stationary points, namely a minimum at $(5, 0)$, a maximum at $(-5, 0)$, and saddles at $(3, 4)$ and $(-3, -4)$.

Selected answers and hints.

1. (a) There is a saddle at $(-2, 12)$.
   (b) There is a maximum at $(0, 0)$.
   (c) There are maxima whenever xy is an even multiple of π, and minima whenever xy is an odd multiple of π. ($\Delta = 0$ for the stationary points here, so thinking about how the function behaves is necessary.)
2. (b) There are no stationary points on this surface.
   (c) Each slice through the surface parallel to the x-axis is a cosine wave, and moving in the positive y-direction, the amplitude of the wave decreases.
3. (a) $e^x \ln(1 + y) \approx y + xy - \frac{y^2}{2} + \dots$
   (b) The next term is
   $$\frac{1}{3!}\left(x^3 f_{xxx}(0,0) + 3x^2 y f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)\right).$$

For more details, start a thread on the discussion board.