PARTIAL DIFFERENTIATION

5 minute review. Remind students that $z = f(x, y)$ represents a surface, how to calculate $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$, and how the partial derivatives relate to gradients on slices through surfaces. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ when $f(x, y) = x^3y^2 - 5e^{xy}$, with help from the students.

Class warm-up. Let $f(x, y) = (x^2 - 1)^2 + \ln(xy) - \frac{1}{2}y^2$. Find the points where $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$. (You could briefly mention the notion of a stationary point: this will be covered in the next video.)

Problems. Choose from the below.

1. **Partial differentiation practice**. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ where
   (a) $f(x, y) = x + y + (x + y)^2 + e^{xy^2}$;
   (b) $f(x, y) = \cos x \tan y - \ln \sqrt{x^2 + y^2}$;
   (c) $f(x, y) = \frac{y}{x} + 2^{xy} - \cosh(x^{-2})$;
   (d) $f(x, y) = \frac{x^3 - 3xy + 1}{e^{x+y}}$.
2. **Chain Rule**. There is a *chain rule* for partial differentiation, which states that if $z = f(x, y)$, where x and y are functions of u and v, then
$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u},$$
   (and similarly for $\frac{\partial z}{\partial v}$, by replacing all u's with v's).
   (a) Let $z = x^4 + 2x^2y^2 + y^4$, where $x = r\cos\theta$ and $y = r\sin\theta$. Use the chain rule to show that $\frac{\partial z}{\partial r} = 4r^3$ and $\frac{\partial z}{\partial \theta} = 0$.
   (b) Check your answers above by expressing z in terms of r and θ and calculating the same derivatives explicitly.
3. **More chain rule**.
   (a) Let $z = \sin(x + y) - \frac{1}{\sqrt{x^2 - y^2}}$, where $x = u + v$ and $y = u - v$. Using the chain rule, show that
$$\frac{\partial z}{\partial u}\frac{\partial z}{\partial v} = \left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2. \tag{1}$$
   (b) Now let $z = f(x, y)$ be any function of x and y, where $x = u + v$ and $y = u - v$. Show that equation (1) again holds.
4. **Contours**.
   (a) What does the surface $z = \sqrt{x^2 + y^2}$ look like? (Hint: think about the *contours*; that is, the points where $z = 0$, $z = 1$, $z = 2$ etc.)
   (b) What about $z = x^2 + y^2$?
   (c) What about $z = \sin(\sqrt{x^2 + y^2})$?

For the review, $\frac{\partial f}{\partial x} = 3x^2y^2 - 5ye^{xy}$ and $\frac{\partial f}{\partial y} = 2x^3y - 5xe^{xy}$.

For the warm-up, $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$ at $(\frac{1}{\sqrt{2}}, 1)$, $(\frac{1}{\sqrt{2}}, -1)$, $(-\frac{1}{\sqrt{2}}, 1)$ and $(-\frac{1}{\sqrt{2}}, -1)$, although only the first and fourth points lie in the domain of f.

Selected answers and hints.

1. (a) $\frac{\partial f}{\partial x} = 1 + 2x + 2y + y^2e^{xy^2}$ and $\frac{\partial f}{\partial y} = 1 + 2x + 2y + 2xye^{xy^2}$.

   (b) $\frac{\partial f}{\partial x} = -\sin(x)\tan(y) - \frac{x}{x^2+y^2}$ and $\frac{\partial f}{\partial y} = \cos(x)\sec^2(y) - \frac{y}{x^2+y^2}$.

   (c) $\frac{\partial f}{\partial x} = y2^{xy}\ln 2 - \frac{y}{x^2} + \frac{2}{x^3}\sinh(x^{-2})$ and $\frac{\partial f}{\partial y} = \frac{1}{x} + x2^{xy}\ln 2$.

   (d) $\frac{\partial f}{\partial x} = \frac{-x^3+3x^2+3xy-3y-1}{e^{x+y}}$ and $\frac{\partial f}{\partial y} = \frac{-x^3+3xy-3x-1}{e^{x+y}}$.

2. (b) Here, it turns out that $z = r^4$, from which the derivatives are clear.

3. (a) Here,

$$\left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2 = \frac{1 + 2(x+y)\cos(x+y)\sqrt{x^2-y^2}}{(x^2-y^2)^2}$$

   and

$$\frac{\partial z}{\partial u}\frac{\partial z}{\partial v} = \frac{1 + 8u\sqrt{uv}\cos(2u)}{16u^2v^2}.$$

4. (a) The contours here are equally-spaced concentric circles. The surface is a cone, with tip at $(0, 0)$.

   (b) This time we get a *paraboloid*.

   (c) This is best described as ripples in a pond. Use Wolfram Alpha to investigate.

For more details, start a thread on the discussion board.