

L'HÔPITAL'S RULE

5 minute review. Recap how to use l'Hôpital's Rule for limits of the form "0/0" (e.g. $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$) and " ∞/∞ " (e.g. $\lim_{x \rightarrow \infty} \frac{x^2+1}{e^x}$). Also cover how sometimes rearrangement or taking logarithms is necessary first (e.g. $\lim_{x \rightarrow \infty} e^{-x} \ln x = 0$ or $\lim_{x \rightarrow \infty} (1 + \frac{1}{x})^x = e$).

Class warm-up. Compute $\lim_{x \rightarrow 0} \frac{\cos x - 1 + \frac{x^2}{2}}{x^4}$ using l'Hôpital's Rule. Check that you get the same answer by replacing $\cos x$ with its Maclaurin series, simplifying the expression and putting $x = 0$.

Problems. Choose from the below.

1. **Standard exercises.** Find the following limits.

(a) $\lim_{h \rightarrow 0} \frac{e^h - 1}{h}$

(b) $\lim_{h \rightarrow 0} \frac{\cos h - 1}{h}$

(c) $\lim_{h \rightarrow 0} \frac{\tan h}{h(1 - a \tan h)}$, where $a \in \mathbb{R}$

2. **Rational functions.**

(a) Find the limits below. (l'Hôpital's Rule may or may not be useful.)

(i) $\lim_{x \rightarrow \infty} \frac{3x^2 + 2}{x^3 - 2x^2 + 1}$

(ii) $\lim_{x \rightarrow \infty} \frac{3x^3 + 2x - 5}{2x^2 + 2}$

(iii) $\lim_{x \rightarrow \infty} \frac{2x^2 + 1}{5x^2 - 3}$

(b) Let $m, n > 0$. What is $\lim_{x \rightarrow \infty} \frac{a_n x^n + \dots + a_1 x + a_0}{b_m x^m + \dots + b_1 x + b_0}$ in each of the cases $n > m$, $n = m$ and $n < m$?

3. **Standard factorisations.** Since $a^2 - b^2 = (a + b)(a - b)$, we have $a - b = \frac{a^2 - b^2}{a + b}$. Use similar techniques to compute the limits below.

(a) $\lim_{x \rightarrow \infty} \sqrt{x^2 + 3x + 3} - x$;

(b) $\lim_{x \rightarrow \infty} \sqrt{x^3 - x^2} - 7x$;

(c) $\lim_{x \rightarrow \infty} \sqrt[4]{x^4 + ax^3} - x$ (where $a > 0$ is any constant).

4. Calculate $\lim_{x \rightarrow 3} \frac{1}{\sqrt{x+1}-2} - \frac{4}{x-3}$.

For the first part of the review, $\lim_{x \rightarrow \infty} e^{-x} \ln x = \lim_{x \rightarrow \infty} \frac{\ln x}{e^x} = -\lim_{x \rightarrow \infty} 1xe^{-x} = 0$, by l'Hôpital. For the second part, consider instead $\lim_{x \rightarrow \infty} \ln \left(1 + \frac{1}{x}\right)^x = \lim_{x \rightarrow \infty} x \ln \left(1 + \frac{1}{x}\right)$. Writing $y = 1/x$, we can rewrite this as $\lim_{y \rightarrow 0} \frac{\ln(1+y)}{y}$. By l'Hôpital, this is $\lim_{y \rightarrow 0} \frac{1}{1+y} = 1$. So the limit of the original function is $e^1 = e$.

For the warm-up, $\lim_{x \rightarrow 0} \frac{\cos x - 1 + \frac{x^2}{2}}{x^4} = \frac{1}{24}$

Selected answers and hints.

1. (a) 1; (b) 0; (c) 1. (These limits appear in differentiating e^x , $\cos x$ and $\tan x$ from first principles.)

2. (a) (i) 0; (ii) ∞ ; (iii) $\frac{2}{5}$.

(b) The general result is that $\lim_{x \rightarrow \infty} \frac{a_n x^n + \dots + a_1 x + a_0}{b_m x^m + \dots + b_1 x + b_0}$ is

- ∞ if $n > m$ and $\frac{a_n}{b_m} > 0$,
- $-\infty$ if $n > m$ and $\frac{a_n}{b_m} < 0$,
- 0 if $m > n$, and
- $\frac{a_n}{b_m}$ if $m = n$.

3. (a) With $a = \sqrt{x^2 + 3x + 3}$ and $b = x$, we get

$$\begin{aligned} \lim_{x \rightarrow \infty} \sqrt{x^2 + 3x + 3} - x &= \lim_{x \rightarrow \infty} \frac{(\sqrt{x^2 + 3x + 3})^2 - x^2}{\sqrt{x^2 + 3x + 3} + x} \\ &= \lim_{x \rightarrow \infty} \frac{3x + 3}{\sqrt{x^2 + 3x + 3} + x} \\ &= \lim_{x \rightarrow \infty} \frac{3 + 3/x}{\sqrt{1 + 3/x + 3/x^2} + 1} = \frac{3}{2}. \end{aligned}$$

(b) Using the technique above, $\lim_{x \rightarrow \infty} \sqrt{x^3 - x^2} - 7x = \infty$;

(c) Applying the suggested technique twice,

$$\begin{aligned} \lim_{x \rightarrow \infty} \sqrt[4]{x^4 + ax^3} - x &= \lim_{x \rightarrow \infty} \frac{\sqrt{x^4 + ax^3} - x^2}{\sqrt[4]{x^4 + ax^3} + x} \\ &= \lim_{x \rightarrow \infty} \frac{x^4 + ax^3 - x^4}{(\sqrt[4]{x^4 + ax^3} + x)(\sqrt{x^4 + ax^3} + x^2)} \\ &= \lim_{x \rightarrow \infty} \frac{ax^3}{(\sqrt[4]{x^4 + ax^3} + x)(\sqrt{x^4 + ax^3} + x^2)} \\ &= \lim_{x \rightarrow \infty} \frac{a}{(\sqrt[4]{1 + \frac{a}{x}} + 1)(\sqrt{1 + \frac{a}{x}} + 1)} \\ &= \frac{a}{4}. \end{aligned}$$

Note that in the penultimate step, dividing top and bottom by x^3 , we divide the first bracket in the denominator by x and the second by x^2 . There is also an alternative, simpler approach using the binomial theorem ($\sqrt[4]{x^4 + ax^3} - x = x(1 + \frac{a}{x})^{\frac{1}{4}} - x = \dots$).

4. By writing the expression as one fraction, and rationalising the denominator, $\lim_{x \rightarrow 3} \frac{1}{\sqrt{x+1}-2} - \frac{4}{x-3} = \frac{1}{4}$.

For more details, start a thread on the discussion board.