PARAMETRIC, IMPLICIT AND LOGARITHMIC DIFFERENTIATION

5 minute review. (This includes the warm-up.) Recap the theory for

- parametric differentiation, with an example like $y = t\sin(t)$, $x = t\cos(t)$ (including a graph);
- implicit differentiation, with an example like $y^2 + xe^y = x^2$;
- how taking logarithms can simplify differentiation, perhaps with $y = \frac{\sin^2(x)e^x}{(x+1)^3}$.

Problems. Choose from the below.

1. **Even more differentiation practice**. Find $\frac{dy}{dx}$ for each of the following curves, simplifying your answer as much as possible.

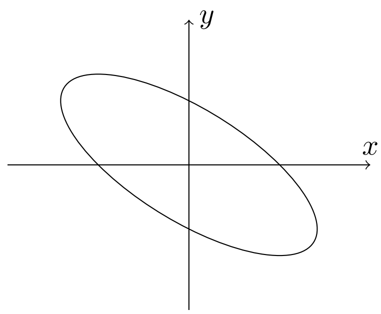
   (a) $x = \frac{2t}{1+t^2}$, $y = \frac{1-t^2}{1+t^2}$ (where $t \in \mathbb{R}$); (b) $x^2 + y^2 - 6xy + 3x - 2y = -5$; (c) $e^y = \frac{(2x^2+1)\cos(x)}{\ln(x+1)}$; (d) $y = \frac{\ln(x)}{2^x}$.

2. Let $x = t - \sin t$, $y = 1 - \cos t$. Show that $\frac{d^2y}{dx^2} = -\frac{1}{y^2}$, where $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$ is the second derivative.

3. The curve defined by the equation $x^2 + 2xy + 2y^2 = 1$ is pictured below.

   (a) Find the point on the curve with the largest x-coordinate.

   (b)* Show that the maximum distance between the curve and the origin is $\frac{1}{2}(\sqrt{5}+1)$. (Hints: The distance s, of a point (x, y) from the origin is $s = \sqrt{x^2+y^2}$. You need to maximise this. Later on in the calculation it may be a good idea to work out $(\frac{1}{2}(\sqrt{5}+1))^2$.)



4. **Differentiating iterates***.

   (a) Let f_n be the function formed by applying the exponential function n times, where n is some positive integer (so that $f_1(x) = e^x$, $f_2(x) = e^{e^x}$, $f_3(x) = e^{e^{e^x}}$, etc). Find $f_1'(x)$, $f_2'(x)$ and $f_3'(x)$. What is the derivative of $f_n(x)$?

   (b) Let g_n be the function formed by applying the natural logarithm function n times, where n is some positive integer (so that $g_1(x) = \ln(x)$, $g_2(x) = \ln(\ln(x))$, $g_3(x) = \ln(\ln(\ln(x)))$, etc). Find $g_1'(x)$, $g_2'(x)$ and $g_3'(x)$. What is the derivative of $g_n(x)$?

For the review, we have the following:

- We get $\frac{dy}{dt} = \sin(t) + t\cos(t)$ and $\frac{dx}{dt} = \cos(t) - t\sin(t)$, and hence
$$\frac{dy}{dx} = \frac{\sin(t) + t\cos(t)}{\cos(t) - t\sin(t)}.$$
- We find $2y\frac{dy}{dx} + e^y + xe^y\frac{dy}{dx} = 2x$, which rearranges to give
$$\frac{dy}{dx} = \frac{2x - e^y}{2y + xe^y}.$$
- We have $\ln y = 2\ln\sin(x) + x - \ln(x+1)^3$, which gives
$$\frac{1}{y}\frac{dy}{dx} = \frac{2\cos x}{\sin x} + 1 - \frac{3}{x+1},$$
and hence
$$\frac{dy}{dx} = \frac{2\sin x\cos x e^x}{(x+1)^3} + \frac{\sin^2(x)e^x}{(x+1)^3} - \frac{3\sin^2(x)e^x}{(x+1)^4}.$$

Selected answers and hints.

1. (a) $\frac{dy}{dx} = \frac{2t}{(t+1)(t-1)}$;
   (b) $\frac{dy}{dx} = \frac{6y-2x-3}{2y-6x-2}$ (as long as $2y - 6x - 2 \neq 0$);
   (c) $\frac{dy}{dx} = \frac{4x}{2x^2+1} - \tan x - \frac{1}{(x+1)\ln(x+1)}$;
   (d) $\frac{dy}{dx} = \frac{1-x\ln 2\ln x}{x2^x}$.
2. Given $\frac{dy}{dx}$ in terms of t, the chain rule says $\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}(\frac{dy}{dx})}{\frac{dx}{dt}}$.
3. (a) The point with the largest (positive) x-coordinate is $(\sqrt{2}, -\frac{1}{\sqrt{2}})$. (Find $\frac{dy}{dx}$ using implicit differentiation and look for the points where the gradient is infinite/undefined.)
   (b) You should find that $\frac{ds}{dx} = 0$ leads to the equation $x^2+xy-y^2 = 0$, which is satisfied whenever $x = \left(-\frac{1}{2} \pm \frac{\sqrt{5}}{2}\right) y$ (using the quadratic formula).

   The original equation $x^2 + 2xy + 2y^2 = 1$ now gives $x^2 = 1 \mp \frac{2\sqrt{5}}{5}$ and $y^2 = \frac{1}{2} \mp \frac{\sqrt{5}}{10}$, which leads to $s = \sqrt{\frac{3}{2} \mp \frac{\sqrt{5}}{2}}$. Thus the maximum value is $s = \sqrt{\frac{3}{2} + \frac{\sqrt{5}}{2}}$, which turns out to be $\frac{1}{2}(\sqrt{5}+1)$. (You can see this final point by squaring $\frac{1}{2}(\sqrt{5}+1)$.)
4. (a) $f_2'(x) = e^{e^x}.e^x = f_2(x).f_1(x)$;
   $f_3'(x) = e^{e^{e^x}}.e^{e^x}.e^x = f_3(x).f_2(x).f_1(x)$;
   $f_n'(x) = f_n(x).f_{n-1}(x).\dots.f_1(x)$.
   (b) $g_2'(x) = \dfrac{1}{x\ln x} = \dfrac{1}{x.g_1(x)}$;
   $g_3'(x) = \dfrac{1}{x.\ln x.\ln(\ln x)} = \dfrac{1}{x.g_1(x).g_2(x)}$;
   $g_n'(x) = \dfrac{1}{x.g_1(x).g_2(x).\dots.g_{n-1}(x)}$.

For more details, start a thread on the discussion board.