

PARAMETRIC, IMPLICIT AND LOGARITHMIC DIFFERENTIATION

5 minute review. (This includes the warm-up.) Recap the theory for

- parametric differentiation, with an example like $y = t \sin(t)$, $x = t \cos(t)$ (including a graph);
- implicit differentiation, with an example like $y^2 + xe^y = x^2$;
- how taking logarithms can simplify differentiation, perhaps with $y = \frac{\sin^2(x)e^x}{(x+1)^3}$.

Problems. Choose from the below.

1. **Even more differentiation practice.** Find $\frac{dy}{dx}$ for each of the following curves, simplifying your answer as much as possible.

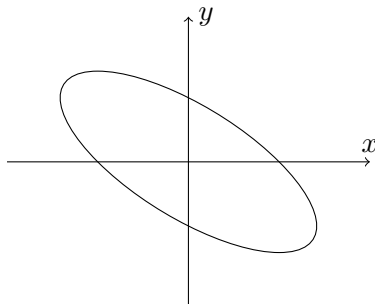
- (a) $x = \frac{2t}{1+t^2}$, $y = \frac{1-t^2}{1+t^2}$ (where $t \in \mathbb{R}$); (b) $x^2 + y^2 - 6xy + 3x - 2y = -5$;
(c) $e^y = \frac{(2x^2+1)\cos(x)}{\ln(x+1)}$; (d) $y = \frac{\ln(x)}{2^x}$.

2. Let $x = t - \sin t$, $y = 1 - \cos t$. Show that $\frac{d^2y}{dx^2} = -\frac{1}{y^2}$, where $\frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right)$ is the second derivative.

3. The curve defined by the equation $x^2 + 2xy + 2y^2 = 1$ is pictured below.

(a) Find the point on the curve with the largest x -coordinate.

(b)* Show that the maximum distance between the curve and the origin is $\frac{1}{2}(\sqrt{5} + 1)$. (Hints: The distance s , of a point (x, y) from the origin is $s = \sqrt{x^2 + y^2}$. You need to maximise this. Later on in the calculation it may be a good idea to work out $(\frac{1}{2}(\sqrt{5} + 1))^2$.)



4. **Differentiating iterates***.

(a) Let f_n be the function formed by applying the exponential function n times, where n is some positive integer (so that $f_1(x) = e^x$, $f_2(x) = e^{e^x}$, $f_3(x) = e^{e^{e^x}}$, etc). Find $f'_1(x)$, $f'_2(x)$ and $f'_3(x)$. What is the derivative of $f_n(x)$?

(b) Let g_n be the function formed by applying the natural logarithm function n times, where n is some positive integer (so that $g_1(x) = \ln(x)$, $g_2(x) = \ln(\ln(x))$, $g_3(x) = \ln(\ln(\ln(x)))$, etc). Find $g'_1(x)$, $g'_2(x)$ and $g'_3(x)$. What is the derivative of $g_n(x)$?

For the review, we have the following:

- We get $\frac{dy}{dt} = \sin(t) + t \cos(t)$ and $\frac{dx}{dt} = \cos(t) - t \sin(t)$, and hence

$$\frac{dy}{dx} = \frac{\sin(t) + t \cos(t)}{\cos(t) - t \sin(t)}.$$

- We find $2y \frac{dy}{dx} + e^y + xe^y \frac{dy}{dx} = 2x$, which rearranges to give

$$\frac{dy}{dx} = \frac{2x - e^y}{2y + xe^y}.$$

- We have $\ln y = 2 \ln \sin(x) + x - \ln(x+1)^3$, which gives

$$\frac{1}{y} \frac{dy}{dx} = \frac{2 \cos x}{\sin x} + 1 - \frac{3}{x+1},$$

and hence

$$\frac{dy}{dx} = \frac{2 \sin x \cos x e^x}{(x+1)^3} + \frac{\sin^2(x) e^x}{(x+1)^3} - \frac{3 \sin^2(x) e^x}{(x+1)^4}.$$

Selected answers and hints.

- (a) $\frac{dy}{dx} = \frac{2t}{(t+1)(t-1)}$;
 - (b) $\frac{dy}{dx} = \frac{6y-2x-3}{2y-6x-2}$ (as long as $2y - 6x - 2 \neq 0$);
 - (c) $\frac{dy}{dx} = \frac{4x}{2x^2+1} - \tan x - \frac{1}{(x+1) \ln(x+1)}$;
 - (d) $\frac{dy}{dx} = \frac{1-x \ln 2 \ln x}{x^{2^x}}$.
- Given $\frac{dy}{dx}$ in terms of t , the chain rule says $\frac{d^2y}{dx^2} = \frac{d}{dt} \left(\frac{dy}{dx} \right) \frac{dt}{dx}$.
- (a) The point with the largest (positive) x -coordinate is $(\sqrt{2}, -\frac{1}{\sqrt{2}})$. (Find $\frac{dy}{dx}$ using implicit differentiation and look for the points where the gradient is infinite/undefined.)
 - (b) You should find that $\frac{ds}{dx} = 0$ leads to the equation $x^2 + xy - y^2 = 0$, which is satisfied whenever $x = \left(-\frac{1}{2} \pm \frac{\sqrt{5}}{2}\right) y$ (using the quadratic formula).

The original equation $x^2 + 2xy + 2y^2 = 1$ now gives $x^2 = 1 \mp \frac{2\sqrt{5}}{5}$ and $y^2 = \frac{1}{2} \mp \frac{\sqrt{5}}{10}$, which leads to $s = \sqrt{\frac{3}{2} \mp \frac{\sqrt{5}}{2}}$. Thus the maximum value is $s = \sqrt{\frac{3}{2} + \frac{\sqrt{5}}{2}}$, which turns out to be $\frac{1}{2}(\sqrt{5} + 1)$. (You can see this final point by squaring $\frac{1}{2}(\sqrt{5} + 1)$.)

- (a) $f_2'(x) = e^{e^x} \cdot e^x = f_2(x) \cdot f_1(x)$;
 $f_3'(x) = e^{e^{e^x}} \cdot e^{e^x} \cdot e^x = f_3(x) \cdot f_2(x) \cdot f_1(x)$;
 $f_n'(x) = f_n(x) \cdot f_{n-1}(x) \cdot \dots \cdot f_1(x)$.
 - (b) $g_2'(x) = \frac{1}{x \ln x} = \frac{1}{x \cdot g_1(x)}$;
 $g_3'(x) = \frac{1}{x \cdot \ln x \cdot \ln(\ln x)} = \frac{1}{x \cdot g_1(x) \cdot g_2(x)}$;
 $g_n'(x) = \frac{1}{x \cdot g_1(x) \cdot g_2(x) \cdot \dots \cdot g_{n-1}(x)}$.

For more details, start a thread on the discussion board.