INVERSE FUNCTION RULE AND CURVE SKETCHING

5 minute review. Briefly cover the inverse trigonometric functions and their derivatives (and remind students that that $\sin^{-1} x$ does not mean $\frac{1}{\sin x}$).

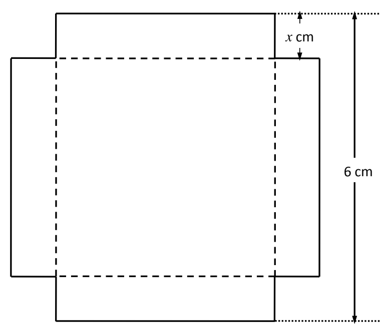
Class warm-up. Run through the proof that $\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$ with input from the class, asking for justification on the sign of the square-root in particular.

Problems. Choose from the below.

1. **More differentiation practice**. Differentiate the following, simplifying your answer as much as possible.

   (a) $f(x) = \cos(\sin x)$; (b) $f(x) = \ln(3x\cos^2(2x))$; (c) $f(x) = \ln \frac{\sin x + \cos x}{\sin x - \cos x}$.

2. **Box revisited**. In week 1 we looked at an open box, made by folding a square piece of card as shown in the figure.

   Prove using calculus that the maximum volume of this box occurs at $x = 1$.

3. **Differentiating** $\operatorname{sech}^{-1} x$.

   (a) Differentiate $y = \operatorname{sech} x$ $(= \frac{1}{\cosh x})$.

   (b) Sketch the graph of $y = \cosh x$ and use this to sketch $y = \operatorname{sech} x$. Choose a domain on which $\operatorname{sech} x$ is invertible. Hence sketch the graph of $y = \operatorname{sech}^{-1} x$ and state its domain and range.

   (c) Using the inverse function rule, show that if $y = \operatorname{sech}^{-1} x$ then
   $$\frac{dy}{dx} = -\frac{1}{x\sqrt{1-x^2}}.$$
   (Hint: use $\cosh^2 x - \sinh^2 x = 1$ to find a relation linking $\tanh^2 x$ and $\operatorname{sech}^2 x$.)

4. **Graph sketching**. Sketch the graph of $y = f(x)$ for each of the following functions. You should think about domains, ranges, stationary points and crossings and ensure that you label all important points.

   (a) $f(x) = \dfrac{x}{(1+2x)^2}$; (b) $f(x) = \left(\dfrac{x-3}{x^2+1}\right)^{\frac{1}{2}}$; (c) $f(x) = \left(\dfrac{x-3}{x^2-1}\right)^{\frac{1}{2}}$.

Selected answers and hints.

1. (a) $f'(x) = -\cos x.\sin(\sin x)$; (b) $f'(x) = \frac{1}{x} - 4\tan(2x)$; (c) $f'(x) = 2\sec(2x)$.
2. Find a formula for the volume, $V(x)$, differentiate, find the stationary point(s) and check that $x = 1$ gives a maximum.
3. (a) $\frac{d}{dx}(\operatorname{sech} x) = -\tanh x \operatorname{sech} x$.
4. (a) $f(x) = \frac{x}{(1+2x)^2}$ is zero when $x = 0$, has an asymptote at $x = -\frac{1}{2}$ and a maximum at $(\frac{1}{2}, \frac{1}{8})$;

   (b) $f(x) = \left(\frac{x-3}{x^2+1}\right)^{\frac{1}{2}}$ has domain $[3, \infty)$, is zero when $x = 3$ and has a maximum at $x = 3 + \sqrt{10}$;

   (c) $f(x) = \left(\frac{x-3}{x^2-1}\right)^{\frac{1}{2}}$ has domain $(-1, 1) \cup [3, \infty)$, has vertical asymptotes at $x = \pm 1$, crosses the y-axis at $y = \sqrt{3}$, is zero when $x = 3$ and has a maximum at $x = 3 + 2\sqrt{2}$ and a minimum at $x = 3 - 2\sqrt{2}$.

For more details, start a thread on the discussion board.