

## DIFFERENTIATION 1: FIRST RESULTS

**5 minute review.** Recall the definition  $\frac{df}{dx}(x_0) = \lim_{h \rightarrow 0} \frac{f(x_0+h) - f(x_0)}{h}$  by working out the gradient of the line on the curve between  $(x_0, f(x_0))$  and  $(x_0+h, f(x_0+h))$ .

**Class warm-up.** Find the derivatives of some of the following functions.

$$f(x) = x^3; g(x) = x^4; h(x) = x^n; j(x) = \frac{1}{x^2}; k(x) = \frac{1}{x^n}.$$

**Problems.** Choose from the below.

1. **Differentiating  $\sqrt{x}$ .** Use the binomial theorem to show that

$$\sqrt{x_0 + h} \approx \sqrt{x_0} + h/(2\sqrt{x_0})$$

when  $h \ll x_0$ . Use the definition above to find the derivative of  $\sqrt{x}$ .

2. **Differentiating  $\tan x$ .** Using the addition formulae for sin and cos, write down an expression for  $\tan(A+B)$  in terms of sines and cosines of  $A$  and  $B$ .

Put  $A = x_0$  and  $B = h$ , where  $h$  is very small, so that  $\sin h \approx h$  and  $\cos h \approx 1$ . Deduce that

$$\tan(x_0 + h) \approx \frac{\tan x_0 + h}{1 - h \tan x_0}.$$

Hence work out the derivative of  $\tan x$  at  $x = x_0$ .

3. **The natural logarithm\***. Put  $f(x) = a^x$ . Recall from the videos that

$$\lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h} = a^{x_0} \lim_{h \rightarrow 0} \frac{a^h - 1}{h},$$

and that  $\lim_{h \rightarrow 0} \frac{a^h - 1}{h}$  turns out to be  $\ln(a)$ . Let's check this.

- (a) First take  $a = 2$ . Use your calculator to work out  $\frac{a^h - 1}{h}$  for  $h = 0.1, 0.01, 0.001$  and  $0.0001$ . How do these values compare to  $\ln 2$ ?

- (b) Now let  $\ell(a) = \lim_{h \rightarrow 0} \frac{a^h - 1}{h}$ . Use small values of  $h$  to check that  $\ell(e)$  appears to be 1.

- (c) Show that  $\frac{(ab)^h - 1}{h} = b^h \frac{a^h - 1}{h} + \frac{b^h - 1}{h}$  and hence that  $\ell(ab) = \ell(a) + \ell(b)$ . This means that  $\ell(a)$  is a logarithm function to some base. Conclude by part (b) that the base is  $e$ .

4. **Some others.** Differentiate the following functions, by any method whatsoever.

(a)  $x^{1000}$ , (b)  $x^5 + x + 1$ , (c)  $e^{2x}$ , (d)  $\ln(2x)$ , (e)  $\sin(-5x)$ .

**Selected answers and hints.**

1. As is well known,  $\frac{d}{dx}(\sqrt{x}) = \frac{1}{2}x^{-\frac{1}{2}}$ .
2. The derivative of  $\tan x$  is well known... use Google!
3. Covered in the video Standard Derivatives 2.
4. (a)  $1000x^{999}$ , (b)  $5x^4 + 1$ , (c)  $2e^{2x}$ , (d)  $\frac{1}{x}$ , (e)  $-5\cos(-5x)$ .

For more details, start a thread on the discussion board.