DIFFERENTIATION 1: FIRST RESULTS

5 minute review. Recall the definition $\frac{df}{dx}(x_0) = \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}$ by working out the gradient of the line on the curve between $(x_0, f(x_0))$ and $(x_0+h, f(x_0+h))$.

Class warm-up. Find the derivatives of some of the following functions.

$$f(x) = x^3;\ g(x) = x^4;\ h(x) = x^n;\ j(x) = \tfrac{1}{x^2};\ k(x) = \tfrac{1}{x^n}.$$

Problems. Choose from the below.

1. **Differentiating $\sqrt{x}$.** Use the binomial theorem to show that

   $$\sqrt{x_0 + h} \approx \sqrt{x_0} + h/(2\sqrt{x_0})$$

   when $h \ll x_0$. Use the definition above to find the derivative of $\sqrt{x}$.

2. **Differentiating** $\tan x$. Using the addition formulae for sin and cos, write down an expression for $\tan(A + B)$ in terms of sines and cosines of A and B.

   Put $A = x_0$ and $B = h$, where h is very small, so that $\sin h \approx h$ and $\cos h \approx 1$. Deduce that

   $$\tan(x_0 + h) \approx \frac{\tan x_0 + h}{1 - h \tan x_0}.$$

   Hence work out the derivative of $\tan x$ at $x = x_0$.

3. **The natural logarithm*.** Put $f(x) = a^x$. Recall from the videos that

   $$\lim_{h\to 0} \frac{f(x_0 + h) - f(x_0)}{h} = a^{x_0} \lim_{h\to 0} \frac{a^h - 1}{h},$$

   and that $\lim_{h\to 0} \frac{a^h-1}{h}$ turns out to be $\ln(a)$. Let's check this.

   (a) First take $a = 2$. Use your calculator to work out $\frac{a^h-1}{h}$ for $h = 0.1$, 0.01, 0.001 and 0.0001. How do these values compare to $\ln 2$?

   (b) Now let $\ell(a) = \lim_{h\to 0} \frac{a^h-1}{h}$. Use small values of h to check that $\ell(e)$ appears to be 1.

   (c) Show that $\dfrac{(ab)^h - 1}{h} = b^h \dfrac{a^h - 1}{h} + \dfrac{b^h - 1}{h}$ and hence that $\ell(ab) = \ell(a) + \ell(b)$. This means that $\ell(a)$ is a logarithm function to some base. Conclude by part (b) that the base is e.

4. **Some others**. Differentiate the following functions, by any method whatsoever.

   (a) x^{1000}, (b) $x^5 + x + 1$, (c) e^{2x}, (d) $\ln(2x)$, (e) $\sin(-5x)$.

Selected answers and hints.

1. As is well known, $\frac{d}{dx}(\sqrt{x}) = \frac{1}{2}x^{-\frac{1}{2}}$.
2. The derivative of $\tan x$ is well known... use Google!
3. Covered in the video Standard Derivatives 2.
4. (a) $1000x^{999}$, (b) $5x^4 + 1$, (c) $2e^{2x}$, (d) $\frac{1}{x}$, (e) $-5\cos(-5x)$.

For more details, start a thread on the discussion board.