GENERAL LOGARITHMS AND HYPERBOLIC FUNCTIONS

5 minute review. Remind students

- what $\log_a x$ is for general $a > 0$ (where $a \neq 1$): that is, $\log_a x$ is the power of a needed to make x, and that $\ln = \log_e$;
- the definitions of sinh, cosh and tanh;
- the identities $\cosh^2(x) - \sinh^2(x) = 1$, $\cosh(x+y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y)$ and $\sinh(x+y) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y)$, emphasising the similarities and differences with trigonometry.

Class warm-up. Prove the identity $\cosh^2(x) - \sinh^2(x) = 1$, with input from the class. Encourage them to tell you to work from the definitions, ask for input on how to structure the solution (start with the left, aim for the right rather than simplifying in an attempt to reach $0 = 0$) etc. You could discuss backwards reasoning in general terms (notes on the course webpage).

Problems. Choose from the below.

1. **Some logarithm calculations**. Without using a calculator, evaluate

$$\text{(a) } \log_9 81; \quad \text{(b) } \log_{12} \tfrac{1}{144}; \quad \text{(c) } \log_2 \tfrac{\sqrt{2}}{8}; \quad \text{(d) } \log_{0.5} 8.$$

2. **Change of base formula**. Let x and a be positive real numbers and let $y = \log_a x$. Writing this as $x = a^y$, take $\log_b$ of both sides to prove the change of base formula,

$$\log_a x = \frac{\log_b x}{\log_b a}.$$

   Hence calculate $\log_7(73)$ using only the ln button on your calculator.

3. **Trigonometrics to hyperbolics**. For each of the following *trigonometric* identities, derive the corresponding hyperbolic statement.

   (a) $\sin^2 x + \cos^2 x = 1$; (b) $\sin(2x) = 2\cos x \sin x$;
   (c) $\cos(2x) = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$;
   (d) $\tan(2x) = \dfrac{2\tan x}{1 - \tan^2 x}$;
   (e) $\cos x \cos y = \frac{1}{2}\cos(x-y) + \frac{1}{2}\cos(x+y)$.

4. **Inverse hyperbolic functions***.

   (a) Suppose that $x > 1$. What is $(x + \sqrt{x^2-1})(x - \sqrt{x^2-1})$? Deduce that $x - \sqrt{x^2-1} < 1$.

   (b) Sketch $y = \cosh x = \frac{1}{2}(e^x + e^{-x})$. Identify a suitable domain on which cosh is invertible, and find an expression for its inverse function. Part (a) will be useful here.

Selected answers and hints.

1. (a) $\log_9 81 = 2$;
   (b) $\log_{12} \frac{1}{144} = -2$;
   (c) $\log_2 \frac{\sqrt{2}}{8} = \log_2 \sqrt{2} - \log_2 8 = \frac{1}{2} - 3 = -\frac{5}{2}$;
   (d) $\log_{0.5} 8 = \log_{0.5}((\frac{1}{2})^{-3}) = -3$.
2. $\log_7(73) = \ln(73)/\ln(7) \approx 2.205$ (3dp).
3. All easy to calculate. For the final part, try adding the formulae for $\cosh(x+y)$ and $\cosh(x-y)$.
4. (a) $(x+\sqrt{x^2-1})(x-\sqrt{x^2-1}) = 1$ by the difference of two squares formula, $(a-b)(a+b) = a^2-b^2$. If $x > 1$ then $(x+\sqrt{x^2-1}) > 1$, so $(x-\sqrt{x^2-1}) < 1$.
   (b) When $\cosh(x)$ has domain $[0,\infty)$, it is an invertible function. Writing $y = \frac{1}{2}(e^x + e^{-x})$ we find that $e^{2x} - 2ye^x + 1 = 0$. This is a quadratic in e^x with solution $e^x = y \pm \sqrt{y^2-1}$, which then gives $x = \ln(y \pm \sqrt{y^2-1})$. Since we must have $x \geq 0$, it follows from part (a) that we need to take the positive root, and hence end up with $\cosh^{-1}(x) = \ln(x + \sqrt{x^2-1})$.

For more details, start a thread on the discussion board.