FUNCTIONS 2: CIRCULAR FUNCTIONS AND BINOMIAL EXPANSION

Announcement. Please encourage students to seek extra help from MASH.

5 minute review. Recap the *multi-angle* or *addition* formulae,

$$\cos(\theta_1 \pm \theta_2) = \cos(\theta_1)\cos(\theta_2) \mp \sin(\theta_1)\sin(\theta_2),$$
$$\sin(\theta_1 \pm \theta_2) = \sin(\theta_1)\cos(\theta_2) \pm \sin(\theta_2)\cos(\theta_1).$$

Recap the binomial coefficients, $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$, and the binomial theorem for integer and non-integer powers.

Class warm-up. Write down the first four terms of the binomial expansion of

$$f(x) = (4-3x)^{\frac{3}{2}}.$$

Be careful to first get this in the correct form for using the binomial theorem (by factoring out the 4). Also, take care with the minus sign.

For what vales of x does the expansion converge? See how well the first four terms approximate $f(x)$ if x is near zero.

Problems. Choose from the below.

1. **Sums and differences**. Expand $R\sin(x+\alpha)$ using the multi-angle formulae, and hence find formulae for $\cos x + \sin x$ and $\cos x - \sin x$ in terms of a single use of sin.
2. **Multi-angle formulae**. Newton's second law of motion for a damped pendulum can be written in the form
$$x(t) = [A\sin\omega t + B\cos\omega t]\, e^{-kt},$$
for some positive constants A, B, ω and k.

   By expanding $C\cos(\omega t - \phi)$, show that $x(t)$ can be written as
$$x(t) = Ce^{-kt}\cos(\omega t - \phi),$$
and find expressions for C and ϕ in terms of A and B. Why must ϕ be in the range $0 \leq \phi \leq \dfrac{\pi}{2}$? Sketch $x(t)$ for $t > 0$.
3. **Approximating roots**. Sketch the curves of the functions
$$f(x) = 2x \quad \text{and} \quad g(x) = (1-x)^{1/3},$$
for $0 \leq x \leq 1$.
   (a) Use the first two terms of the binomial expansion of $g(x)$ to find an approximation x_1 to the x-value of the point of intersection of the two curves.
   (b) Find another approximation x_2 by using the first three terms of the expansion.
   (c) Add the curves of your binomial expansions to your sketch. Given that the true value of x at the point of intersection of $f(x)$ and $g(x)$ is $x_* = 0.4176$ (to 4dp), what is the percentage error of the two approximations?

For the warm-up, the answer is

$$f(x) = 8 - 9x + \frac{27}{16}x^2 + \frac{27}{128}x^3 + \dots$$

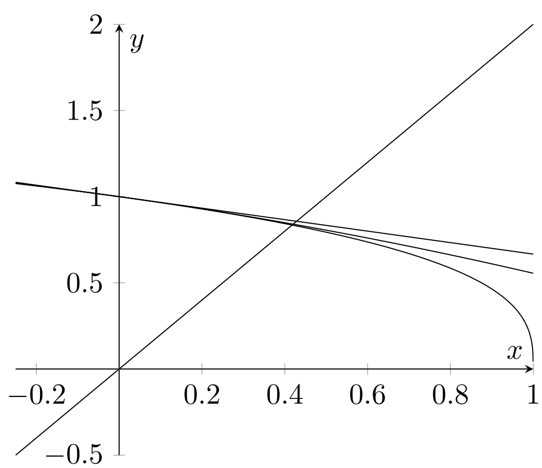
which will converge for $|\frac{3}{4}x| < 1$, i.e. $|x| < 4/3$.

Selected answers and hints.

1. We have
$$\cos x + \sin x = \sqrt{2} \sin\left(x + \frac{\pi}{4}\right),$$
and
$$\cos x - \sin x = -\sqrt{2} \sin\left(x - \frac{\pi}{4}\right).$$

2. $C\cos(\omega t - \phi) = C(\cos(\omega t)\cos\phi + \sin(\omega t)\sin\phi)$. If we can find C and ϕ such that
$$C\sin\phi = A \quad (1)$$
$$C\cos\phi = B \quad (2)$$
then we're done. Squaring and adding, $A^2+B^2 = C^2(\sin^2\phi+\cos^2\phi) = C^2$, so $C = \sqrt{A^2+B^2}$. To find ϕ, dividing (1) by (2) gives $\tan\phi = \frac{A}{B}$. Since A and B are both positive, we must have $\sin\phi$ and $\cos\phi$ both positive, and hence ϕ be in the range $0 \le \phi \le \frac{\pi}{2}$. It follows that $\phi = \tan^{-1}(\frac{A}{B})$. (In general, one might need a different solution for ϕ, for example $\tan^{-1}(A/B) + \pi$.)

3. (a) $g(x) \approx 1 - \frac{1}{3}x$, which gives $x_1 = \frac{3}{7} \approx 0.4286$ (4dp) as the x-value at the intersection.
   (b) $g(x) \approx 1 - \frac{1}{3}x - \frac{1}{9}x^2$, which gives $x_2 \approx 0.4202$ (4dp) as the x-value at the intersection.
   (c) The curves are shown below.



For more details, start a thread on the discussion board.

About MASH. Mathematics and Statistics Help (MASH) is a free service run by the University at the 301 Study Centre (301 Glossop Road, opposite Bar One). Their aim is to help students who do maths as part of their degree, but are not maths students. They run drop-in sessions and you can book appointments.

For more information, see their website: `http://shef.ac.uk/mash`.