FUNCTIONS 2: CIRCULAR FUNCTIONS AND BINOMIAL EXPANSION

5 minute review. Recap the *multi-angle* or *addition* formulae,

$$\cos(\theta_1 \pm \theta_2) = \cos(\theta_1)\cos(\theta_2) \mp \sin(\theta_1)\sin(\theta_2),$$
$$\sin(\theta_1 \pm \theta_2) = \sin(\theta_1)\cos(\theta_2) \pm \sin(\theta_2)\cos(\theta_1).$$

Recap the binomial coefficients, $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$, and the binomial theorem for integer and non-integer powers.

Class warm-up. Write down the first four terms of the binomial expansion of

$$f(x) = (4 - 3x)^{\frac{3}{2}}.$$

Be careful to first get this in the correct form for using the binomial theorem (by factoring out the 4). Also, take care with the minus sign.

For what vales of x does the expansion converge? See how well the first four terms approximate $f(x)$ if x is near zero.

Problems. Choose from the below.

1. **Sums and differences**. Expand $R\sin(x+\alpha)$ using the multi-angle formulae, and hence find formulae for $\cos x + \sin x$ and $\cos x - \sin x$ in terms of a single use of sin.

2. **Approximating roots**. Sketch the curves of the functions

   $$f(x) = 2x \quad \text{and} \quad g(x) = (1-x)^{1/3},$$

   for $0 \le x \le 1$.

   (a) Use the first two terms of the binomial expansion of $g(x)$ to find an approximation x_1 to the x-value of the point of intersection of the two curves.

   (b) Find another approximation x_2 by using the first three terms of the expansion.

   (c) Add the curves of your binomial expansions to your sketch. Given that the true value of x at the point of intersection of $f(x)$ and $g(x)$ is $x_* = 0.4176$ (to 4dp), what is the percentage error of the two approximations?

3. **Product-to-sum formulae**.

   (a) Using the addition and subtraction formulae for sin, verify the identity

   $$\sin\alpha\cos\beta = \tfrac{1}{2}\sin(\alpha+\beta) + \tfrac{1}{2}\sin(\alpha-\beta).$$

   (b) Using the addition and subtraction formulae for cos, find similar identities for $\sin\alpha\sin\beta$ and for $\cos\alpha\cos\beta$.

   (c) Use these identities to write $\sin^2(x)$ and then $\sin^3(x)$ as sums of individual trigonometric functions.

For the warm-up, the answer is

$$f(x) = 8 - 9x + \frac{27}{16}x^2 + \frac{27}{128}x^3 + \dots$$

which will converge for $|\frac{3}{4}x| < 1$, i.e. $|x| < 4/3$.

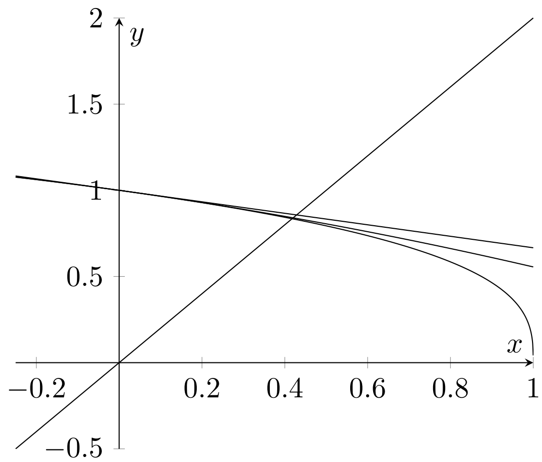
Selected answers and hints.

1. We have
$$\cos x + \sin x = \sqrt{2}\sin\left(x + \frac{\pi}{4}\right),$$
and
$$\cos x - \sin x = -\sqrt{2}\sin\left(x - \frac{\pi}{4}\right).$$

2. (a) $g(x) \approx 1 - \frac{1}{3}x$, which gives $x_1 = \frac{3}{7} \approx 0.4286$ (4dp) as the x-value at the intersection.

   (b) $g(x) \approx 1 - \frac{1}{3}x - \frac{1}{9}x^2$, which gives $x_2 \approx 0.4202$ (4dp) as the x-value at the intersection.

   (c) The curves are shown below.



3. The other two formulae are
$$\sin\alpha\sin\beta = \tfrac{1}{2}\cos(\alpha - \beta) - \tfrac{1}{2}\cos(\alpha + \beta),$$
$$\cos\alpha\cos\beta = \tfrac{1}{2}\cos(\alpha + \beta) + \tfrac{1}{2}\cos(\alpha - \beta).$$

   The resulting formulae for powers are

$\sin^2(x) = \frac{1}{2}\cos(0) - \frac{1}{2}\cos(2x) = \frac{1}{2}(1 - \cos(2x)),$

$\sin^3(x) = \frac{1}{2}\sin(x)(1 - \cos(2x)) = \frac{1}{2}\sin(x) - \frac{1}{2}\sin(x)\cos(2x) = \frac{3}{4}\sin(x) - \frac{1}{4}\sin(3x).$

   These identities are particularly useful for integrating trigonometric functions!

For more details, start a thread on the discussion board.