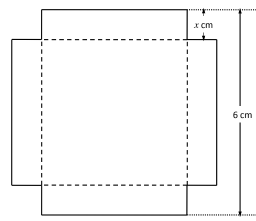
RANGE, PARITY AND PERIODICITY

5 minute review. Recap the definition of a function, and the meaning of domain, range, even/odd functions and periodicity.

Warm-up. An open box is made by folding a square piece of card of side 6cm as shown in the figure.

Find a formula for the volume $V(x)$ of the box. What is an appropriate domain for this function? Sketch $V(x)$ over this domain. What can you say about the range of $V(x)$? Show that $V(1) = 16\text{cm}^3$. Write $x = (1+\epsilon)$cm for $-1 < \epsilon < 2$, and rearrange the resulting expression for V to deduce that the maximum volume is achieved when $\epsilon = 0$ (i.e. $x = 1$cm).



(Do not allow the use of calculus!)

Problems. Choose from the below.

1. **Exploring functions**.
   (a) Are the following functions even, odd, or neither? What can you say about their range?
   $$\text{(i) } \frac{3x}{x^2+2}, \quad \text{(ii) } \cos x + 3, \quad \text{(iii) } \frac{2x^3}{\cos x + 3}, \quad \text{(iv) } \frac{3x^4+2x^2}{\sin x + 4}.$$
   (b) Consider the functions $\sin^n x$ and $\cos^n x$, where n is a positive integer. Which are odd, which are even, and what are their periodicities?
   (c) What is the periodicity of $f(x) = 2\cos x + \sin 2x$? Is it even/odd/neither?
2. **Cubic functions**. Show that $x - 1$ is a factor of the cubic function
   $$P(x) = x^3 - 2x^2 - 5x + 6,$$
   and find the quadratic factor $Q(x)$ such that $P(x) = (x-1)Q(x)$. Factorise $Q(x)$ and hence sketch $P(x)$.
3. **Sigmoid functions***. For positive integers n, consider the family of functions
   $$f_n(x) = \frac{x^n}{2^n + x^n}.$$
   (a) What are appropriate domains and ranges for $f_n(x)$? Do they depend on n?
   (b) Is $f_n(x)$ even, odd, or neither? Does this depend on n?
   (c) Show that for $x \geq 0$, $f_n(x)$ are strictly increasing functions and that $0 \leq f_n(x) < 1$.
   (d) What is the value of $f_n(2)$? Does it depend on n?
   (e) By considering the values of $f_n(1.9)$ and $f_n(2.1)$ for $n = 2, 4, 8, 16, 32$ and 64, show that the graph of $f_n(x)$ for $x \geq 0$ is an increasing sigmoid (that is, S-shaped) function that approaches a step function as $n \to \infty$.

For the warm-up, $V(x) = 4x(3-x)^2$. Writing $x = 1+\epsilon$, we get

$$V = 4(1+\epsilon)(3-(1+\epsilon))^2 = 4(4-\epsilon^2(3-\epsilon)).$$

If $-1 < \epsilon < 2$ then $\epsilon^2(3-\epsilon) \geq 0$, with equality when $\epsilon = 0$. Thus the maximum value of V occurs when $\epsilon = 0$, i.e. $x = 1$cm.

Selected answers and hints.

1. (a) (i) Odd. The range is the interval $[-\frac{3}{4}\sqrt{2}, +\frac{3}{4}\sqrt{2}]$.
   (ii) Even. The range is the interval $[2,4]$.
   (iii) Odd. The range is all of $\mathbb{R}$.
   (iv) Neither. (For example, writing $f(x) = \frac{3x^4+2x^2}{\sin x+4}$ we have $f(1) = \frac{5}{\sin(1)+4} \approx 1.032$ (3dp) whereas $f(-1) = \frac{5}{\sin(-1)+4} \approx 1.583$ (3dp), which is neither $f(1)$ nor $-f(1)$.)

   The range is the nonnegative reals $[0,\infty)$.

   (b) For any n, $\cos^n(-x) = (\cos(-x))^n = (\cos x)^n = \cos^n x$, so $\cos^n x$ is even. On the other hand, $\sin^n(-x) = (\sin(-x))^n = (-\sin x)^n = (-1)^n \sin^n x$. Thus $\sin^n x$ is even if n is even and odd if n is odd. For odd n, both $\sin^n x$ and $\cos^n x$ have period 2π. For even n, their periods are both π.

   (c) Since $\cos x$ has period 2π and $\sin(2x)$ has period π, it follows that $f(x) = 2\cos x + \sin(2x)$ repeats every 2π. It doesn't repeat more frequently. (To see this, you could find where it crosses the x-axis, perhaps.) It is neither odd nor even:

   $$f(-x) = 2\cos(-x) + \sin(2(-x)) = 2\cos(x) - \sin(2x),$$

   which is neither $f(x)$ or $-f(x)$.

   (To be thorough, we should show that $f(-x)$ isn't a rearrangement of $\pm f(x)$ by giving a value where it's clearly different. For example $f(\pi/4) = 2.\frac{1}{\sqrt{2}} + 1 = \sqrt{2}+1$, whereas $f(-\pi/4) = 2.\frac{1}{\sqrt{2}} - 1 = \sqrt{2}-1$, which is neither $\pm f(\pi/4)$.)

2. The factor theorem says that if $f(x)$ is a polynomial with $f(a) = 0$, then $(x-a)$ is a factor of $f(x)$. In our case, $P(1) = 1-2-5+6 = 0$, so $(x-1)$ is a factor of $P(x)$. Here, $P(x) = (x-1)(x-3)(x+2)$, so $Q(x) = (x-3)(x+2)$, so $P(x)$ has roots at -2, 1 and 3.

3. (a) The domain of $f_n(x)$ is $\mathbb{R}$ if n is even, and $x \in \mathbb{R}$, $x \neq -2$ for n odd.

   (b) If n is even then $f_n(x)$ is even; if n is odd then $f_n(x)$ is neither odd nor even.

   (d) $f_n(2) = \frac{1}{2}$ for all n.

For more details, start a thread on the discussion board.