MAS156: Mathematics (Electrical and Aerospace)

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Tuesday 16th October 2018, 1pm
Diamond LT4

Course matters

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Copies of exams from previous years are also on the site.

Matrices

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Why matrices?

Matrices as transformations

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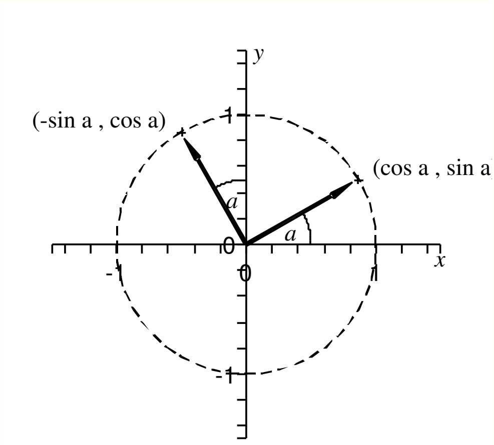

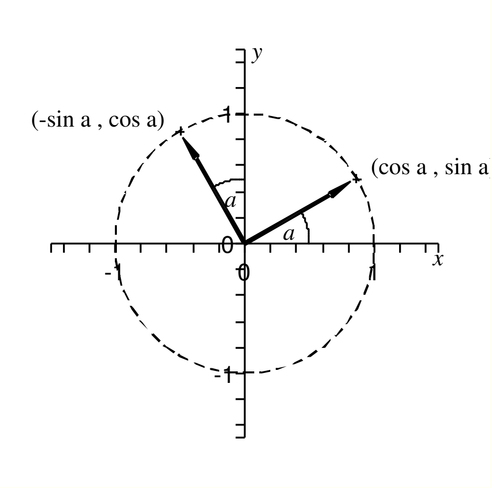

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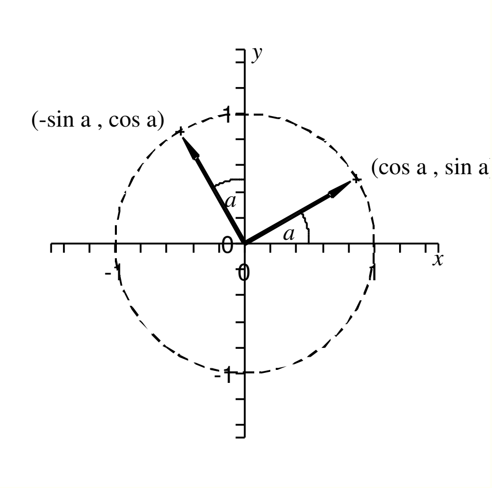

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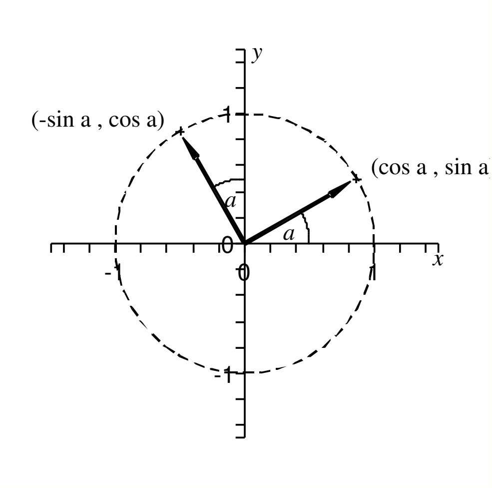

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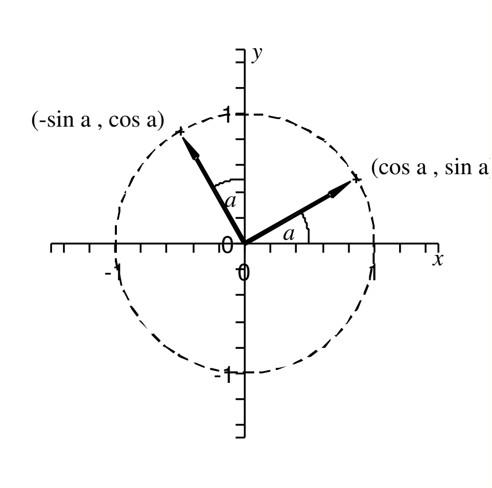

Using geometry, we find that the point $(1, 0)$ transforms to $(\cos a, \sin a)$ and $(0, 1)$ transforms to $(-\sin a, \cos a)$.

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Once again, problems like these are best solved using matrices.

Definitions

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We sometimes write $A = (a_{ij})$ for the above matrix.

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is called the *identity matrix of size* n.

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The identity matrix I_n is always *square*. That is, it has the same number of rows and columns.

Matrix operations

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$$A + B = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \dots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \dots & a_{2n} + b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \dots & a_{mn} + b_{mn} \end{pmatrix}.$$

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In other words, to add two matrices *of the same dimensions* simply add their entries componentwise.

For example,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 2 & 0 & 3 \\ 4 & 2 & 0 \end{pmatrix} =$$

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$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 2 & 0 & 3 \\ 4 & 2 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 3 \\ 4 & 3 & 0 \end{pmatrix}.$$

Warning!

It is not possible to add two matrices if their dimensions are different, so take care!

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In other words, to multiply a matrix by a scalar, k, simply multiply each entry of the matrix by k.

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$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 & 1 \\ 2 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1.1 + 0.2 & 1.3 + 0.0 & 1.1 + 0.(-1) \\ 0.1 + 1.2 & 0.3 + 1.0 & 0.1 + 1.(-1) \end{pmatrix}$$

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Activity. Working in groups of two or three, in each case find a matrix A such that

(i) $$A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\cos a - y\sin a \\ x\sin a + y\cos a \end{pmatrix}.$$

(ii) $$A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x - 2y + z \\ 2y - 8z \\ -4x + 5y + 9z \end{pmatrix}.$$

(iii) $$A\begin{pmatrix} x_{\mathsf{urban}} \\ x_{\mathsf{suburban}} \end{pmatrix} = \begin{pmatrix} 0.95x_{\mathsf{urban}} + 0.03x_{\mathsf{suburban}} \\ 0.05x_{\mathsf{urban}} + 0.97x_{\mathsf{suburban}} \end{pmatrix}.$$

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This matrix corresponds to rotation of the plane through an angle a: given a point (x, y), calculating

$$A \begin{pmatrix} x \\ y \end{pmatrix}$$

gives the coordinates of where it ends up after the rotation.

(ii) $A = \begin{pmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ -4 & 5 & 9 \end{pmatrix}.$

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Notice that the equations in the example at the start of the lecture correspond to the matrix equation

$$A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 8 \\ -9 \end{pmatrix}.$$

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$$A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 8 \\ -9 \end{pmatrix}.$$

The solution is then

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1} \begin{pmatrix} 0 \\ 8 \\ -9 \end{pmatrix}.$$

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will give the amount of people in the urban and suburban areas after one year.

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In the example at the beginning of the lecture,

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will give the amount of people in the urban and suburban areas after one year. Multiplying by A repeatedly means the populations after 25 years will be given by

$$A^{25} \begin{pmatrix} 600,000 \\ 400,000 \end{pmatrix}.$$

And finally. . .

Reminders:

- email address mas-engineering@sheffield.ac.uk
- website http://engmaths.group.shef.ac.uk/mas156
  (also accessible through MOLE).