

# MAS156: Mathematics (Aerospace)

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Tuesday 22nd October 2019, 1pm  
Alfred Denny LT2

# Course matters

**Your comments**

We are interested to know your opinions about this course:

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If you have comments about your class tutor, please mention them by name.

**Reading week**

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You should use the time to revise or catch up with the material so far, e.g by working on exercises.

# Matrices

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**Why matrices?**

# Matrices to model systems

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A certain city consists of an urban area and suburbs. Each year 5% of those living in the urban area move to the suburbs and 3% of those living in the suburbs move to the urban area. If there are initially 600,000 people in the urban area and 400,000 in the suburbs, how many are in each 25 years later?

# Definitions

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We sometimes write  $A = (a_{ij})$  for the above matrix.

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is called the *identity matrix of size  $n$* .

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# Matrix operations

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In other words, to add two matrices *of the same dimensions* simply add their entries componentwise.

For example,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 2 & 0 & 3 \\ 4 & 2 & 0 \end{pmatrix} =$$

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$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 2 & 0 & 3 \\ 4 & 2 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 3 \\ 4 & 3 & 0 \end{pmatrix}.$$

# Warning!

It is not possible to add two matrices if their dimensions are different, so take care!

# Matrix multiplication

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# Column vectors

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**Activity.** Working in groups of two or three, find a matrix  $A$  such that

$$A \begin{pmatrix} x_{\text{urban}} \\ x_{\text{suburban}} \end{pmatrix} = \begin{pmatrix} 0.95x_{\text{urban}} + 0.03x_{\text{suburban}} \\ 0.05x_{\text{urban}} + 0.97x_{\text{suburban}} \end{pmatrix}.$$

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In the example at the beginning of the lecture,

$$A \begin{pmatrix} 600,000 \\ 400,000 \end{pmatrix}$$

will give the number of people in the urban and suburban areas after one year.

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will give the number of people in the urban and suburban areas after one year. Multiplying by  $A$  repeatedly means the populations after 25 years will be given by

$$A^{25} \begin{pmatrix} 600,000 \\ 400,000 \end{pmatrix}.$$

**And finally. . .**

## Reminders:

- email address [mas-engineering@sheffield.ac.uk](mailto:mas-engineering@sheffield.ac.uk)
- website <http://engmaths.group.shef.ac.uk/mas156>  
(also accessible through MOLE).