

MAS153: Mathematics (Materials)

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Friday 25th October 2019, 11am
Stephenson Building LT1

Course matters

Your comments

We are interested to know your opinions about this course:

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If you have comments about your class tutor, please mention them by name.

Reading week

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You should use the time to revise or catch up with the material so far, e.g by working on exercises.

Matrices

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Why matrices?

Matrices to model systems

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A certain city consists of an urban area and suburbs. Each year 5% of those living in the urban area move to the suburbs and 3% of those living in the suburbs move to the urban area. If there are initially 600,000 people in the urban area and 400,000 in the suburbs, how many are in each 25 years later?

Definitions

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We sometimes write $A = (a_{ij})$ for the above matrix.

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is called the *identity matrix of size n* .

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The identity matrix I_n is always *square*. That is, it has the same number of rows and columns.

Matrix operations

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In other words, to add two matrices *of the same dimensions* simply add their entries componentwise.

For example,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 2 & 0 & 3 \\ 4 & 2 & 0 \end{pmatrix} =$$

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$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 2 & 0 & 3 \\ 4 & 2 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 3 \\ 4 & 3 & 0 \end{pmatrix}.$$

Warning!

It is not possible to add two matrices if their dimensions are different, so take care!

Matrix multiplication

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Activity. Working in groups of two or three, find a matrix A such that

$$A \begin{pmatrix} x_{\text{urban}} \\ x_{\text{suburban}} \end{pmatrix} = \begin{pmatrix} 0.95x_{\text{urban}} + 0.03x_{\text{suburban}} \\ 0.05x_{\text{urban}} + 0.97x_{\text{suburban}} \end{pmatrix}.$$

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$$A \begin{pmatrix} 600,000 \\ 400,000 \end{pmatrix}$$

will give the number of people in the urban and suburban areas after one year.

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will give the number of people in the urban and suburban areas after one year. Multiplying by A repeatedly means the populations after 25 years will be given by

$$A^{25} \begin{pmatrix} 600,000 \\ 400,000 \end{pmatrix}.$$

And finally. . .

Reminders:

- email address mas-engineering@sheffield.ac.uk
- website <http://engmaths.group.shef.ac.uk/mas153>
(also accessible through MOLE).