

# MAS151: Civil Engineering Mathematics

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Monday 15th October 2018, 10am  
Dainton Building LT1

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# Course matters

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Copies of exams from previous years are also on the site.

# Matrices

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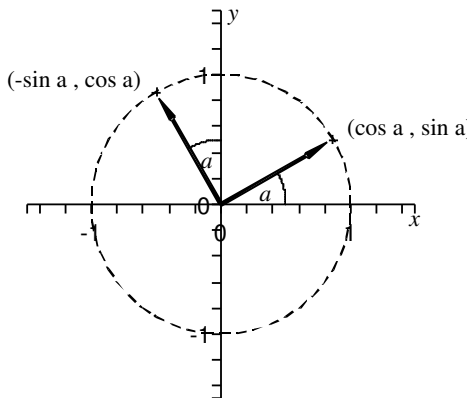
**Why matrices?**

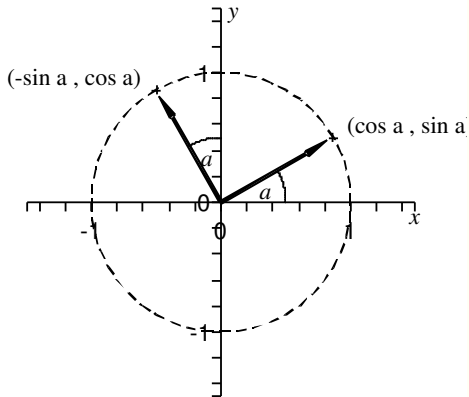
# Matrices as transformations

Let  $0 \leq a < 2\pi$  and consider the transformation of the plane given by anticlockwise rotation through the angle  $a$ , as shown below.

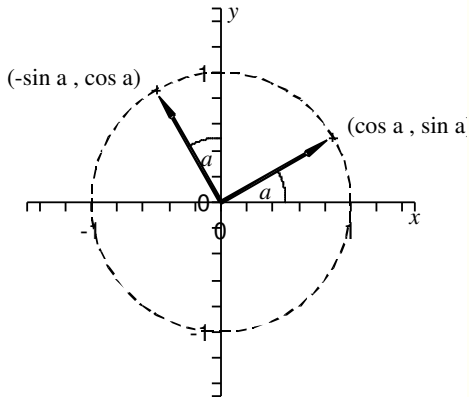
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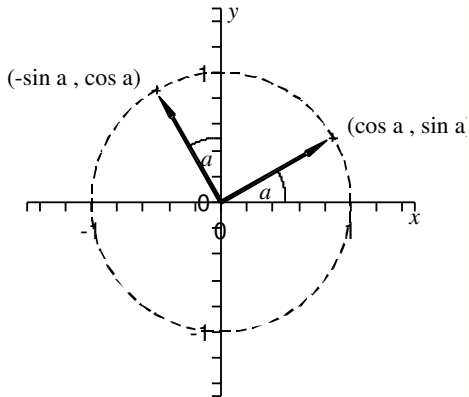




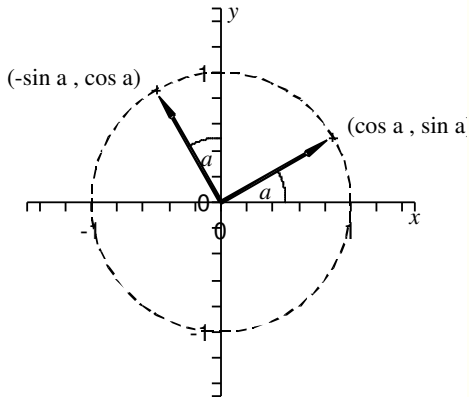
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Using geometry, we find that the point  $(1, 0)$  transforms to  $(\cos a, \sin a)$  and  $(0, 1)$  transforms to  $(-\sin a, \cos a)$ .

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Once again, problems like these are best solved using matrices.

# Definitions

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We sometimes write  $A = (a_{ij})$  for the above matrix.

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is called the *identity matrix of size  $n$* .

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The identity matrix  $I_n$  is always *square*. That is, it has the same number of rows and columns.

# Matrix operations

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$$A + B = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \dots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \dots & a_{2n} + b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \dots & a_{mn} + b_{mn} \end{pmatrix}.$$

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In other words, to add two matrices *of the same dimensions* simply add their entries componentwise.

For example,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 2 & 0 & 3 \\ 4 & 2 & 0 \end{pmatrix} =$$

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# Warning!

It is not possible to add two matrices if their dimensions are different, so take care!

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$$kA = \begin{pmatrix} ka_{11} & ka_{12} & \dots & ka_{1n} \\ ka_{21} & ka_{22} & \dots & ka_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ ka_{m1} & ka_{m2} & \dots & ka_{mn} \end{pmatrix}.$$

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That is, if  $A$  is  $p \times q$  and  $B$  is  $q \times r$ , then we can find their product. The result,  $AB$ , is a  $p \times r$  matrix.

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# Column vectors

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One case that occurs frequently is when the second matrix is a *column vector* (i.e. an  $n \times 1$  matrix) of a suitable length. For example,

$$\begin{pmatrix} 1 & 3 & 1 \\ 2 & 0 & -1 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 1.3 + 3.2 + 1.2 \\ 2.3 + 0.2 + (-1).2 \end{pmatrix} =$$



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**Activity.** Working in groups of two or three, in each case find a matrix  $A$  such that

$$(i) \quad A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \cos a - y \sin a \\ x \sin a + y \cos a \end{pmatrix}.$$

$$(ii) \quad A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x - 2y + z \\ 2y - 8z \\ -4x + 5y + 9z \end{pmatrix}.$$

$$(iii) \quad A \begin{pmatrix} x_{\text{urban}} \\ x_{\text{suburban}} \end{pmatrix} = \begin{pmatrix} 0.95x_{\text{urban}} + 0.03x_{\text{suburban}} \\ 0.05x_{\text{urban}} + 0.97x_{\text{suburban}} \end{pmatrix}.$$

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This matrix corresponds to rotation of the plane through an angle  $a$ : given a point  $(x, y)$ , calculating

$$A \begin{pmatrix} x \\ y \end{pmatrix}$$

gives the coordinates of where it ends up after the rotation.

$$(ii) \quad A = \begin{pmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ -4 & 5 & 9 \end{pmatrix}.$$

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Notice that the equations in the example at the start of the lecture correspond to the matrix equation

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The solution is then

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1} \begin{pmatrix} 0 \\ 8 \\ -9 \end{pmatrix}.$$



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In the example at the beginning of the lecture,

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will give the amount of people in the urban and suburban areas after one year. Multiplying by  $A$  repeatedly means the populations after 25 years will be given by

$$A^{25} \begin{pmatrix} 600,000 \\ 400,000 \end{pmatrix}.$$

**And finally. . .**

## Reminders:

- email address [mas-engineering@sheffield.ac.uk](mailto:mas-engineering@sheffield.ac.uk)
- website <http://engmaths.group.shef.ac.uk/mas151>  
(also accessible through MOLE).