MAS151: Civil Engineering Mathematics

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Friday 25th October 2019, 12noon
Hicks Building LT1

Course matters

Your comments

We are interested to know your opinions about this course:

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If you have comments about your class tutor, please mention them by name.

Reading week

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You should use the time to revise or catch up with the material so far, e.g by working on exercises.

Matrices

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Why matrices?

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A certain city consists of an urban area and suburbs. Each year 5% of those living in the urban area move to the suburbs and 3% of those living in the suburbs move to the urban area. If there are initially $600,000$ people in the urban area and $400,000$ in the suburbs, how many are in each 25 years later?

Definitions

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We sometimes write $A = (a_{ij})$ for the above matrix.

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is called the *identity matrix of size* n.

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The identity matrix I_n is always *square*. That is, it has the same number of rows and columns.

Matrix operations

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In other words, to add two matrices *of the same dimensions* simply add their entries componentwise.

For example,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 2 & 0 & 3 \\ 4 & 2 & 0 \end{pmatrix} =$$

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Warning!

It is not possible to add two matrices if their dimensions are different, so take care!

Matrix multiplication

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&= \begin{pmatrix} 1 & 3 & 1 \\ 2 & 0 & -1 \end{pmatrix}.
\end{aligned}
$$

Activity. Working in groups of two or three, find a matrix A such that

$$A \begin{pmatrix} x_{\textsf{urban}} \\ x_{\textsf{suburban}} \end{pmatrix} = \begin{pmatrix} 0.95x_{\textsf{urban}} + 0.03x_{\textsf{suburban}} \\ 0.05x_{\textsf{urban}} + 0.97x_{\textsf{suburban}} \end{pmatrix}.$$

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$$A \begin{pmatrix} 600,000 \\ 400,000 \end{pmatrix}$$

will give the number of people in the urban and suburban areas after one year.

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will give the number of people in the urban and suburban areas after one year. Multiplying by A repeatedly means the populations after 25 years will be given by

$$A^{25} \begin{pmatrix} 600,000 \\ 400,000 \end{pmatrix}.$$

And finally. . .

Reminders:

- email address mas-engineering@sheffield.ac.uk
- website http://engmaths.group.shef.ac.uk/mas151
  (also accessible through MOLE).