

MAS140: Mathematics (Chemical)

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Wednesday 17th October 2018, 1pm
Diamond LT3

Course matters

There is a Formula Sheet which can be used in exams (it is provided with the exam paper). You will find a copy of this on the webpage.

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Copies of exams from previous years are also on the site.

Matrices

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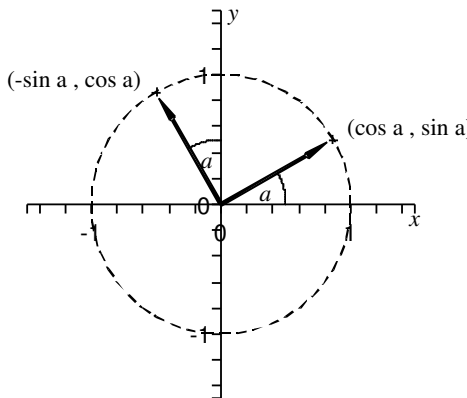
Why matrices?

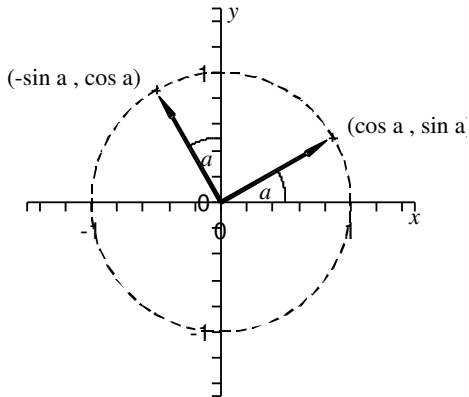
Matrices as transformations

Let $0 \leq a < 2\pi$ and consider the transformation of the plane given by anticlockwise rotation through the angle a , as shown below.

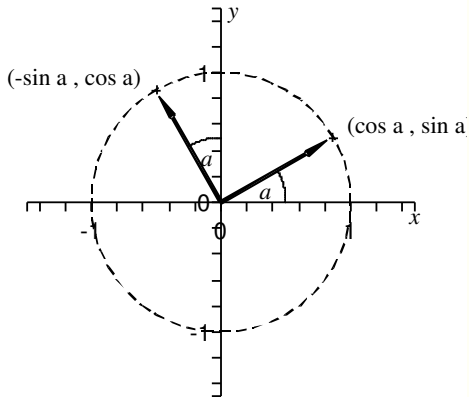
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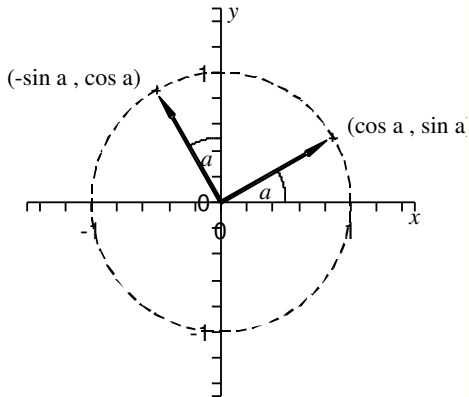




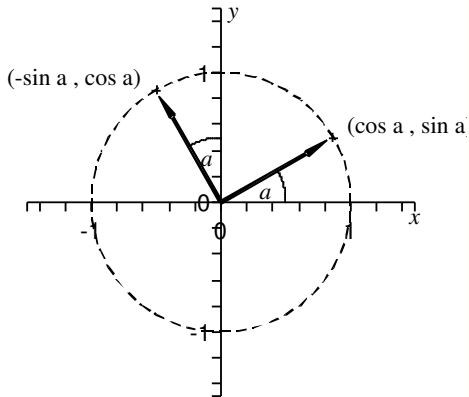
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It turns out that a general point (x, y) transforms to $(x \cos a - y \sin a, x \sin a + y \cos a)$, and this transformation is best described using a matrix.

Matrices to solve equations

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By adding and subtracting multiples of the equations from each other, we find that the solution is $x = 29$, $y = 16$ and $z = 3$. We will later see that there is a systematic approach to solving such systems, again using matrices.

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Once again, problems like these are best solved using matrices.

Definitions

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We sometimes write $A = (a_{ij})$ for the above matrix.

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is called the *identity matrix of size n* .

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The identity matrix I_n is always *square*. That is, it has the same number of rows and columns.

Matrix operations

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In other words, to add two matrices *of the same dimensions* simply add their entries componentwise.

For example,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 2 & 0 & 3 \\ 4 & 2 & 0 \end{pmatrix} =$$

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Warning!

It is not possible to add two matrices if their dimensions are different, so take care!

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Column vectors

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Activity. Working in groups of two or three, in each case find a matrix A such that

$$(i) \quad A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \cos a - y \sin a \\ x \sin a + y \cos a \end{pmatrix}.$$

$$(ii) \quad A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x - 2y + z \\ 2y - 8z \\ -4x + 5y + 9z \end{pmatrix}.$$

$$(iii) \quad A \begin{pmatrix} x_{\text{urban}} \\ x_{\text{suburban}} \end{pmatrix} = \begin{pmatrix} 0.95x_{\text{urban}} + 0.03x_{\text{suburban}} \\ 0.05x_{\text{urban}} + 0.97x_{\text{suburban}} \end{pmatrix}.$$

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This matrix corresponds to rotation of the plane through an angle a : given a point (x, y) , calculating

$$A \begin{pmatrix} x \\ y \end{pmatrix}$$

gives the coordinates of where it ends up after the rotation.

$$(ii) \quad A = \begin{pmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ -4 & 5 & 9 \end{pmatrix}.$$

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Notice that the equations in the example at the start of the lecture correspond to the matrix equation

$$A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 8 \\ -9 \end{pmatrix}.$$

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The solution is then

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1} \begin{pmatrix} 0 \\ 8 \\ -9 \end{pmatrix}.$$

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In the example at the beginning of the lecture,

$$A \begin{pmatrix} 600,000 \\ 400,000 \end{pmatrix}$$

will give the amount of people in the urban and suburban areas after one year.

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In the example at the beginning of the lecture,

$$A \begin{pmatrix} 600,000 \\ 400,000 \end{pmatrix}$$

will give the amount of people in the urban and suburban areas after one year. Multiplying by A repeatedly means the populations after 25 years will be given by

$$A^{25} \begin{pmatrix} 600,000 \\ 400,000 \end{pmatrix}.$$

And finally. . .

Reminders:

- email address mas-engineering@sheffield.ac.uk
- website <http://engmaths.group.shef.ac.uk/mas140>
(also accessible through MOLE).