LAPLACE TRANSFORMS AND DIFFERENTIAL EQUATIONS WITH REVISION

5 minute review. Recap the Laplace transform and the differentiation rule, and observe that this gives a good technique for solving linear differential equations: translating them to algebraic equations, and handling the initial conditions.

Class warm-up. Find a solution to the differential equation

$$\frac{dy}{dx} - 3y = e^{3x}$$

such that $y = 1$ when $x = 0$.

Problems. (Choose from the below)

I. **Inverse Laplace transforms**. Use the method of partial fractions where necessary to find the inverse Laplace transforms $f(t)$, $g(t)$ and $h(t)$ of the following:

$$F(s) = \frac{s+3}{s^2+6s+25}, \qquad G(s) = \frac{6}{s^2-s-2}, \qquad H(s) = \frac{2}{s^3+s^2+s+1}.$$

II. **A first-order example**. Solve the following differential equation using the Laplace transform:

$$\frac{dy}{dx} = xe^x + 2e^x + y, \qquad \text{where } y = 3 \text{ when } x = 0.$$

III. **Some second-order examples**. Solve the following differential equation using the Laplace transform:

$$\frac{d^2y}{dx^2} + 9y = 18e^{3x}, \qquad \text{where } y = 0 \text{ and } \frac{dy}{dx} = 1 \text{ when } x = 0;$$

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 6xe^{2x}, \qquad \text{where } y = 1 \text{ and } \frac{dy}{dx} = 2 \text{ when } x = 0.$$

IV. **A system of simultaneous differential equations***.

Solve the following differential equations using the Laplace transform:

$$\frac{dx}{dt} = 4x + y, \qquad \frac{dy}{dt} = 2x + 3y, \qquad x(0) = 2, \qquad y(0) = 5.$$

V. **Multiplying by** $t^\star$. It can be shown that, if $\mathcal{L}(f(t)) = F(s)$, then $\mathcal{L}(tf(t)) = -F'(s)$.

- Deduce from this that $\mathcal{L}(tf'(t)) = -sF'(s) - F(s)$ and $\mathcal{L}(tf''(t)) = f(0) - 2sF(s) - s^2F'(s)$.
- Hence find a solution to the differential equation

$$x\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} - 3y = 0$$

such that $y = 0$ and $\frac{dy}{dx} = 1$ when $x = 0$.

For the warmup, the Laplace transform $Y(s)$ of $y(x)$ satisfies

$$sY(s) - y(0) - 3Y(s) = \frac{1}{s-3}.$$

Substituting in $y(0) = 1$ and rearranging, this means that

$$Y(s) = \frac{1 + \frac{1}{s-3}}{s-3} = \frac{1}{s-3} + \frac{1}{(s-3)^2}.$$

Using the shift rule, the inverse Laplace transform of this is $y(x) = e^{3x} + xe^{3x}$.

Selected answers and hints.

I.

$$F(s) = \frac{s+3}{(s+3)^2 + 4^2}, \qquad \text{so} \quad f(t) = e^{-3t}\cos(4t);$$

$$G(s) = \frac{2}{s-2} - \frac{2}{s+1}, \qquad \text{so} \quad g(t) = 2e^{2t} - 2e^{-t};$$

$$H(s) = \frac{1}{s+1} + \frac{1}{s^2+1} - \frac{s}{s^2+1}, \qquad \text{so} \quad h(t) = e^{-t} + \sin(t) - \cos(t).$$

II. The Laplace transform gives us $sY(s) - 3 = \frac{1}{(s-1)^2} + \frac{2}{s-1} + Y(s)$. Solving gives $Y(s) = \frac{3}{s-1} + \frac{2}{(s-1)^2} + \frac{1}{(s-1)^3}$, whence $y = e^x\left(3 + 2x + \frac{x^2}{2}\right)$.

III. For the first one, we get $s^2Y(s) - 1 + 9Y(s) = \frac{18}{s-3}$, and so $Y(s) = \frac{18}{(s-3)(s^2+9)} + \frac{1}{s^2+9} = \frac{1}{s-3} - \frac{s}{s^2+9} - \frac{2}{s^2+9}$, which means that $y = e^{3x} - \cos(3x) - \frac{2}{3}\sin(3x)$.

For the second one, we get $(s^2F(s) - s - 2) - 4(sF(s) - 1) + 4F(s) = 6/(s-2)^2$, which rearranges to give $F(s) = 1/(s-2) + 6/(s-2)^4$. So the answer is $y = (1 + x^3)e^{2x}$.

IV. The Laplace transforms satisfy

$$sX(s) - 2 = 4X(s) + Y(s), \qquad sY(s) - 5 = 2X(s) + 3Y(s).$$

Rearranging, we get

$$X(s) = \frac{Y(s)+2}{s-4}, \qquad Y(s) = \frac{2X(s)+5}{s-3},$$

and then substituting in, we get

$$X(s) = \frac{\frac{2X(s)+5}{s-3} + 2}{s-4} = \frac{2X(s) + 5 + 2(s-3)}{(s-3)(s-4)},$$

and so

$$X(s) = \frac{\frac{2s-1}{(s-3)(s-4)}}{1 - \frac{2}{(s-3)(s-4)}} = \frac{2s-1}{s^2 - 7x + 10} = \frac{2s-1}{(s-2)(s-5)} = \frac{3}{s-5} - \frac{1}{s-2},$$

and hence, by taking the inverse Laplace transform, $x(t) = 3e^{5t} - e^{2t}$. By a similar process, $y(t) = 3e^{5t} + 2e^{2t}$.

V. The Laplace transform gives $(3-s)F'(s) = 2F(s)$, which is separable with solution $F(s) = \frac{a}{(s-3)^2}$. Hence $y = axe^{3x}$, and using the initial conditions we find $a = 1$.

For more details, start a thread on the discussion board.

Revision

5 minute review.

- *Functions*: curve sketching, binomial theorem, inverse functions, exponential & logarithms, trigonometric & hyperbolic functions;
- *Differentiation*: first principles, differentiation rules, parametric & implicit differentiation, partial differentiation;
- *Series*: Maclaurin & Taylor series, l'Hôpital's rule;
- *Complex numbers*: polar & exponential forms, Argand diagram, Euler's relation, de Moivre's theorem;
- *Vectors*: scalar product, vector product;
- *Integration*: substitution, parts, definite integrals, improper integrals;
- *Matrices*: multiplication, determinants, inverses, systems of equations, eigenvectors;
- *Differential equations*: separation of variables, integrating factors, second-order methods, simultaneous DEs.

Extra Problems.

I. **Functions**. Find the stationary points and sketch the graph of $y = \dfrac{x}{1+x^2}$.

II. **Differentiation**. If $f(x,y) = xy^2\cosh(x^2y)$, find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

III. **Limits**.

(a) Use the binomial theorem to evaluate $\lim_{x\to 0}\frac{\sqrt{9-2x}-3}{x}$.

(b) Evaluate $\lim_{x\to 0}\frac{x-\tan x}{x-\sin x}$.

IV. **Complex numbers**. The complex numbers z_1 and z_2 satisfy $\mathrm{Re}(z_1z_2) = \mathrm{Re}(z_1)\,\mathrm{Re}(z_2)$. What (if anything) can you deduce about z_1 and z_2?

V. **Vectors**. The position vector of a particle at time $t \geq 0$ is given by

$$\mathbf{r} = (6\sin(t^2), 6\cos(t^2), (1+4t)^{3/2}).$$

Find the velocity of the particle at time t and verify that the speed of the particle varies linearly with time.

VI. **Integration**. Compute the indefinite integrals

(a) $\displaystyle\int \frac{3(\arctan x)^2 - 1}{x^2+1}\,dx$;

(b) $\displaystyle\int \frac{\sin x + 2\cos x}{2\sin x + \cos x}\,dx$.

VII. **Matrices**. Find the inverse of the matrix

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Selected answers and hints.

I. We have $\frac{dy}{dx} = (1-x^2)/(1+x^2)^2$, so $\frac{dy}{dx} = 0 \iff x = \pm 1$. Thus the stationary points are at $(1, 0.5)$ (a maximum) and $(-1, -0.5)$ (a minimum). The graph passes through the origin and tends to zero at $\pm\infty$.

II. $\frac{\partial f}{\partial x} = y^2\cosh(x^2y) + 2x^2y^3\sinh(x^2y)$ and $\frac{\partial f}{\partial y} = 2xy\cosh(x^2y) + x^3y^2\sinh(x^2y)$.

III. (a) Firstly, $\sqrt{9-2x} = (9-2x)^{\frac{1}{2}} = 9^{\frac{1}{2}}(1-\frac{2x}{9})^{\frac{1}{2}} = 3(1+\frac{1}{2}(-\frac{2x}{9})+\ldots)$, where all further terms have a factor of x^2. Thus

$$\lim_{x\to 0}\frac{\sqrt{9-2x}-3}{x} = \lim_{x\to 0}\frac{(3-\frac{x}{3}+\ldots)-3}{x} = \lim_{x\to 0}\left(-\frac{1}{3}+\ldots\right) = -\frac{1}{3}.$$

(b) The limit is of the form $\frac{0}{0}$, hence

$$\begin{aligned}
\lim_{x\to 0}\frac{x-\tan x}{x-\sin x} &= \lim_{x\to 0}\frac{1-\sec^2 x}{1-\cos x} \quad \text{(by l'Hôpital's Rule)}\\
&= \lim_{x\to 0}\frac{\cos^2 x - 1}{\cos^2 x - \cos^3 x} \quad \text{(rearranging)}\\
&= \lim_{x\to 0}\frac{-2\cos x\sin x}{-2\cos x\sin x + 3\cos^2 x\sin x} \quad \text{(l'Hôpital again)}\\
&= \lim_{x\to 0}\frac{-2\cos x}{-2\cos x + 3\cos^2 x} \quad \text{(cancelling)}\\
&= -2.
\end{aligned}$$

IV. Writing $z_1 = a+ib$, $z_2 = c+id$ we have $\mathrm{Re}(z_1z_2) = ac - bd$ and $\mathrm{Re}(z_1)\,\mathrm{Re}(z_2) = ac$. Thus $bd = 0$, so at least one of z_1 or z_2 must have zero imaginary part (i.e. is real).

V. The velocity vector is

$$\begin{aligned}
\dot{\mathbf{r}} &= (12t\cos(t^2),\ -12t\sin(t^2),\ (3/2).(1+4t)^{1/2}.4)\\
&= (12t\cos(t^2),\ -12t\sin(t^2),\ 6(1+4t)^{1/2}).
\end{aligned}$$

Thus the speed is given by

$$\begin{aligned}
|\dot{\mathbf{r}}| &= \sqrt{144t^2\cos^2(t^2) + 144t^2\sin^2(t^2) + 36(1+4t)}\\
&= 6\sqrt{(2t+1)^2}\\
&= 6(2t+1),
\end{aligned}$$

which varies linearly with t.

VI. (a) Substituting $u = \tan^{-1}x$, $\int \frac{3(\tan^{-1}x)^2-1}{x^2+1}dx = (\tan^{-1}x)^3 - \tan^{-1}x + c$.

(b) Using the standard substitution $t = \tan(x/2)$ and partial fractions, we find

$$\begin{aligned}
\int\frac{\sin x + 2\cos x}{2\sin x + \cos x}dx &= \int\frac{-4t^2+4t+4}{(1+t^2)(-t^2+4t+1)}dt\\
&= \frac{1}{5}\int\left(\frac{-6t}{1+t^2} + \frac{8}{1+t^2} + \frac{-6t+12}{-t^2+4t+1}\right)dt\\
&= \frac{1}{5}\left(-3\ln(1+t^2) + 8\tan^{-1}t + 3\ln(|-t^2+4t+1|)\right) + c.
\end{aligned}$$

VII. It turns out that $A^{-1} = A^T$.

For more details, start a thread on the discussion board.