INHOMOGENEOUS SECOND-ORDER LINEAR EQUATIONS AND LAPLACE TRANSFORMS

5 minute review. For an inhomogeneous second-order linear differential equation with constant coefficients $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$, remind students that

- the *complementary function* is the general solution y_c to the homogeneous equation $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$;
- the *particular integral* is any solution y_p (found with sensible guesswork);
- the general solution is $y_c + y_p$.

Remind students of the definition of the Laplace transform $F(s) = \mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)dt$, and go over the linearity, shift and differentiation rules. Mention the convention of using uppercase letters for Laplace transforms (F for f, G for g, and so on) and also that a table of standard Laplace transforms will be in the exam formula booklet; it is reproduced at the end of the sheet.

Class warm-up. Call for some sensible suggestions for choices for particular integrals for each $f(x)$ below.

$$x^2 - 3; \quad e^{-5x}; \quad \cos x; \quad \sin(3x); \quad x - e^x; \quad x^{100} + \cos(100x) - e^{100x}.$$

Compute the Laplace transform of $f(t) = 1$ by hand, and hence go over the Laplace transform of $f(t) = t$ (which was in the video).

Problems. (Choose from the below)

I. **Standard examples.**. Find the general solutions to the following differential equations.

$$\frac{d^2y}{dx^2} - y = \sin x; \qquad \frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = 10x^3.$$

II. **Using the rules**. Find, using the results of the formula booklet, the Laplace transforms of:

$$f(t) = 4\cos(2t); \qquad g(t) = t^5 e^{-t}; \qquad h(t) = 5e^{4t}\sin(3t) + 2\cosh(7t).$$

III. **The shift rule**. Check the shift rule for yourselves: in other words, show that, if $\mathcal{L}(f(t)) = F(s)$, then $\mathcal{L}(e^{at}f(t)) = F(s-a)$.

IV. **Transforms of** $\sin$ **and** $\cos$.

(a) Integrate by parts twice (integrating the trigonometric function and differentiating the exponential) to show that

$$\int_0^\infty \sin(t)e^{-st}dt = 1 - s^2 \int_0^\infty \sin(t)e^{-st}dt.$$

(b) Deduce that $\mathcal{L}(\sin(t)) = \frac{1}{1+s^2}$.

(c) Use the differentiation rule to get a formula for $\mathcal{L}(\cos)$.

V. **A fourth-order equation?**. Find the general solution to

$$2\frac{d^4y}{dx^4} + 2\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} = x.$$

Hint: Start by integrating both sides of the equation twice (and remember your constants of integration!).

For the warm-up, sensible choices for y_p are

- $ax^2 + bx + c$;
- Ae^{-5x};
- $a\cos x + b\sin x$;
- $a\sin(3x) + b\cos(3x)$;
- $ax + b + ce^x$;
- $a_{100}x^{100} + a_{99}x^{99} + \ldots + a_1x + a_0 + b\cos(100x) + c\sin(100x) + de^{100x}$.

Some of these suggestions could need modifying by adding in additional powers of x if it turns out that they form part of the complementary function. For example, the suggestion Ae^{-5x} will need to become Axe^{-5x} if -5 is a root of the auxiliary equation, or Ax^2e^{-5x} if -5 is a repeated root.

For the warm-up, when $f(t) = 1$ we have

$$F(s) = \int_0^\infty e^{-st}dt = \left[\frac{e^{-st}}{-s}\right]_0^\infty = 0 - \frac{1}{-s} = \frac{1}{s},$$

and then when $f(t) = t$ we have (by integrating by parts)

$$F(s) = \int_0^\infty te^{-st}dt = \left[t\frac{e^{-st}}{-s}\right]_0^\infty - \int_0^\infty \frac{e^{-st}}{-s}dt = 0 + \frac{1}{s}\int_0^\infty e^{-st}dt,$$

and using the previous result, that's $\frac{1}{s^2}$.

Selected answers and hints.

I. (a) The auxiliary equation is $k^2 - 1 = 0$. For the particular integral, try $y_p = a\sin x + b\cos x$. You should get something equivalent to $y = ae^x + be^{-x} - \frac{1}{2}\sin x$.

(b) The auxiliary equation is $k^2 - 2k + 5 = 0$ and hence the complementary function is $y_c = e^x(a\cos(2x) + b\sin(2x))$. Try $y_p = ax^3 + bx^2 + cx + d$ for the particular integral and hence $y_p = 2x^3 + \frac{12}{5}x^2 - \frac{12}{25}x - \frac{144}{125}$.

II. They are:

$$F(s) = \frac{4s}{s^2+4}; \quad G(s) = \frac{120}{(s+1)^6}; \quad H(s) = \frac{15}{(s-4)^2+9} + \frac{2s}{s^2-49}.$$

III. We have

$$\mathcal{L}(e^{at}f(t)) = \int_0^\infty e^{at}f(t)e^{-st}dt = \int_0^\infty f(t)e^{-(s-a)t}dt = F(s-a).$$

IV. The differentiation rule gives us that $\mathcal{L}(\cos(t)) = \frac{s}{1+s^2}$.

V. By integrating twice, the equation becomes $2\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = \frac{1}{6}x^3 + c_1x + c_2$, which is now a standard second-order case. The auxiliary equation is $2k^2 + 2k + 1 = 0$ which has complex solutions. Choose a third order polynomial for the particular integral. The general solution turns out to be

$$y = e^{-\frac{1}{2}x}(A\cos(x/2) + B\sin(x/2)) + \frac{1}{6}x^3 - x^2 + Cx + D.$$

For more details, start a thread on the discussion board.

Table of Laplace transforms

Function $f(t)$	Laplace transform $F(s)$
t^n	$\dfrac{n!}{s^{n+1}}$ (for $n = 0, 1, 2, \ldots$)
e^{at}	$\dfrac{1}{s-a}$
$\sin \omega t$	$\dfrac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\dfrac{s}{s^2+\omega^2}$
$\sinh \omega t$	$\dfrac{\omega}{s^2-\omega^2}$
$\cosh \omega t$	$\dfrac{s}{s^2-\omega^2}$
$e^{at}f(t)$	$F(s-a)$ (shift theorem)
$f'(t)$	$sF(s) - f(0)$ (differentiation theorem)
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$
$af(t) + bg(t)$	$aF(t) + bG(t)$ (linearity)

Extra Problems.

I. **Standard example**. Find the general solutions to the following differential equation:

$$\frac{d^3y}{dx^3} - y = e^x.$$

II. **A tenth-order equation?**. Find the general solution to $\frac{d^2x}{dt^2} + x = 0$. Hence find a particular solution to

$$\frac{d^{10}x}{dt^{10}} + x = 2050e^{2t}$$

for which $x = 3$ when $t = 0$ and $x = 1$ when $t = \frac{\pi}{2}$.

NB: You are not asked to find the general solution of the tenth-order equation, just any particular solution which works.

III. **Transforms of polynomials**. Continue the warm-up exercise to show that if $f(t) = t^n$, for $n = 2, 3, 4$, then its Laplace transform $F(s) = \mathcal{L}(f(t))$ is given by $F(s) = \frac{n!}{s^{n+1}}$. After these you should believe the general case! *(This can be done directly, by integration by parts, or indirectly, using the differentiation rule.)*

IV. **Hyperbolic functions**.

(a) Find formulae for the Laplace transforms of sinh and cosh, by following the strategy of Problem IV in the main problems.

(b) Find the same formulae directly from the definitions of sinh and cosh, using the linearity rules and the shift rule.

Selected answers and hints.

I. The auxiliary equation is $k^3 - 1 = 0$ and hence the complementary function is $y_c = ae^x + e^{-x/2}(b\cos(\sqrt{3}x/2) + c\sin(\sqrt{3}x/2))$. Since e^x is in the complementary function we need a particular integral of the form axe^x and hence $y_p = \frac{1}{3}xe^x$.

II. The general solution to the second-order equation is $x = A\cos t + B\sin t$. Now, if x is any solution to $\frac{d^2x}{dt^2} + x = 0$, then

- $\frac{d^4x}{dt^4} + \frac{d^2x}{dt^2} = 0$, so $\frac{d^4x}{dt^4} - x = 0$;
- $\frac{d^6x}{dt^6} + \frac{d^4x}{dt^4} = 0$, so $\frac{d^6x}{dt^6} + x = 0$;
- $\frac{d^8x}{dt^8} + \frac{d^6x}{dt^6} = 0$, so $\frac{d^8x}{dt^8} - x = 0$;
- $\frac{d^{10}x}{dt^{10}} + \frac{d^8x}{dt^8} = 0$, so $\frac{d^{10}x}{dt^{10}} + x = 0$.

Hence any solution to $\frac{d^2x}{dt^2} + x = 0$ is also a solution to the homogeneous equation $\frac{d^{10}x}{dt^{10}} + x = 0$. To solve $\frac{d^{10}x}{dt^{10}} + x = 2050e^{2t}$ we need a particular integral, and it turns out that $x = 2e^{2t}$ works. Thus $x = A\cos t + B\sin t + 2e^{2t}$ is a solution to the tenth-order equation. Putting in the initial conditions allows you to find A and B and leads to the particular solution $x = (1 - 2e^{\pi})\sin t + \cos t + 2e^{2t}$, which satisfies the requirements.

III.

IV. Differentiating twice and rearranging, as in Problem 2, gives that

$$\mathcal{L}(\sinh(t)) = \frac{1}{s^2 - 1},$$

and then the differentiation rule gives

$$\mathcal{L}(\cosh(t)) = \frac{s}{s^2 - 1}.$$

Alternatively,

$$\begin{aligned}\mathcal{L}(\sinh(t)) =& \mathcal{L}\left(\frac{e^t - e^{-t}}{2}\right) \quad \text{(by definition of sinh)}\\ =& \tfrac{1}{2}\left(\mathcal{L}(e^t) - \mathcal{L}(e^{-t})\right) \quad \text{(by linearity)}\\ =& \frac{1}{2}\left(\frac{1}{s-1} - \frac{1}{s+1}\right) \quad \text{(by shift)}\\ =& \frac{1}{s^2 - 1},\end{aligned}$$

and a very similar calculation gives the corresponding result for cosh.

For more details, start a thread on the discussion board.