INTEGRATING FACTORS AND HOMOGENEOUS SECOND-ORDER LINEAR EQUATIONS

5 minute review. Remind students how to use integrating factors to solve linear first-order differential equations, namely that if $\frac{dy}{dx} + P(x)y = Q(x)$, then the integrating factor is $I(x) = \exp(\int P(x)dx)$, and multiplying through by $I(x)$ makes the left-hand side into an exact derivative. The example $\frac{dy}{dx} + \frac{y}{x} = x^3$ is a good one, if needed.

For the homogeneous second-order equation $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$, cover how to solve it by looking at the roots of the *auxiliary equation* $ak^2 + bk + c = 0$. Deal with the cases of

- distinct real roots (solution: $y = Ae^{\alpha_1 x} + Be^{\alpha_2 x}$);
- complex roots (solution: $y = e^{\alpha x}(A\cos(\beta x) + B\sin(\beta x))$, where the roots are $\alpha \pm \beta i$; mention Euler's identity: $e^{i\theta} = \cos\theta + i\sin\theta$);
- repeated roots (solution: $y = Ae^{\alpha x} + Bxe^{\alpha x}$).

Class warm-up. Find the general solution to the second-order differential equation $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0$.

Problems. (Choose from the below)

I. **A basic example**. Find the general solution to the differential equation

$$\frac{dy}{dx} + \frac{2y}{x} = \frac{\sin x}{x^2}.$$

II. **An example with initial conditions**. Find the solution of the differential equation

$$\exp\left(\frac{dy}{dx} + \frac{y}{x}\right) = x$$

which satisfies $x = 1$ at $y = 1$.

III. **The three cases**. Solve each of the following homogeneous second-order linear differential equations:

(a) $$\frac{d^2y}{dx^2} - 10\frac{dy}{dx} + 24y = 0,$$

(b) $$\frac{d^2y}{dx^2} - 10\frac{dy}{dx} + 25y = 0,$$

(c) $$\frac{d^2y}{dx^2} - 10\frac{dy}{dx} + 26y = 0.$$

IV. **Derivatives of high order**. Find the general solution to the differential equation $\frac{d^2y}{dx^2} + 4y = 0$ and check that all these solutions are also solutions to the differential equation $\frac{d^6y}{dx^6} + 64y = 0$.

V. **Daily temperature cycles**. Recall Newton's law of cooling, which gives a differential equation describing the temperature of a body T in terms of the ambient temperature T_a, namely

$$\frac{dT}{dt} = -k(T - T_a).$$

In the videos we solved this differential equation when T_a was constant. Instead, let's assume $T_a = \sin t$ (a crude approximation of a daily temperature cycle). Find the solution of the resulting differential equation.

For the review, the integrating factor is x and the solution is $y = \frac{x^4}{5} + \frac{C}{x}$.

For the warm-up, the auxiliary equation is $k^2 + k + 1 = 0$ hence

$$y = e^{-\frac{1}{2}x}\left(a\cos\left(\frac{\sqrt{3}}{2}x\right) + b\sin\left(\frac{\sqrt{3}}{2}x\right)\right).$$

Selected answers and hints.

I. The integrating factor is x^2 and the solution is

$$y = \frac{c - \cos x}{x^2}.$$

II. The integrating factor is x: integration by parts is needed and the solution turns out to be

$$y = \frac{1}{2}x\ln x - \frac{1}{4}x + \frac{5}{4x}.$$

III. You should get something equivalent to this:

(a) Auxiliary equation $k^2 - 10k + 24 = 0$ hence $y = ae^{6x} + be^{4x}$,

(b) Auxiliary equation $k^2 - 10k + 25 = 0$ hence $y = (a + bx)e^{5x}$,

(c) Auxiliary equation $k^2 - 10k + 26 = 0$ hence $y = e^{5x}(a\cos x + b\sin x)$.

IV. The auxiliary equation is $k^2 + 4 = 0$, hence the general solution is $y = a\cos(2x) + b\sin(2x)$.

We could differentiate six times to check this, but it's faster to notice that if we have a solution to $\frac{d^2y}{dx^2} + 4y = 0$, then $\frac{d^2y}{dx^2} = -4y$ and, differentiating repeatedly,

$$\begin{aligned}
\frac{d^3y}{dx^3} &= -4\frac{dy}{dx};\\
\frac{d^4y}{dx^4} &= -4\frac{d^2y}{dx^2} = -4(-4y) = 16y;\\
\frac{d^5y}{dx^5} &= 16\frac{dy}{dx};\\
\frac{d^6y}{dx^6} &= 16\frac{d^2y}{dx^2} = 16(-4y) = -64y,
\end{aligned}$$

so $\frac{d^6y}{dx^6} + 64y = 0$.

V. The integrating factor is (as in the example considered in the videos) e^{kt}, and integration by parts is needed to find $\int e^{kt}\sin t\,dt$. Don't keep integrating for ever! Instead combine terms on the LHS and RHS. The solutions turn out to be

$$T = \frac{k(k\sin t - \cos t)}{k^2 + 1} + ce^{-kt},$$

where c is a constant.

For more details, start a thread on the discussion board.

Extra Problems.

I. **A non-linear equation**. By considering the expansion of $(s-t)(s+t-u)$, find two independent families of solutions to

$$\left(\frac{dy}{dx}\right)^2 - x\frac{dy}{dx} - y^2 + xy = 0.$$

II. **Repeated roots**.

(a) Can you find a 3-parameter family of solutions to

$$\frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + y = 0?$$

(b) Can you construct a fourth-order differential equation with general solution $y = Ae^x + Bxe^x + Cx^2e^x + Dx^3e^x$?

(c) What rule are you discovering here?

Selected answers and hints.

I. The equation factorises as $\left(\frac{dy}{dx} - y\right)\left(\frac{dy}{dx} + y - x\right) = 0$, and we can find one family of solutions for each bracket. The first is solved by separating variables, the second has an integrating factor of e^x. This leads to the solutions $y = Ae^x$ and $y = x - 1 + Be^{-x}$.

II. (a) As the auxiliary equation is $k^3+3k^2+3k+1 = 0$, this is $(k+1)^3 = 0$, so there is repeated root $k = -1$ with multiplicity 3. It turns out that $y = e^{-x}$, $y = xe^{-x}$ and $y = x^2e^{-x}$ are all solutions, so the general solution is $y = ae^{-x} + bxe^{-x} + cx^2e^{-x}$.

(b) Following the pattern in (a) we might expect a repeated root $k = +1$ with multiplicity 4. So $(m-1)^4$ is the auxiliary equation. The given expression will therefore be the general solution to $\frac{d^4y}{dx^4} - 4\frac{d^3y}{dx^3} + 6\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + y = 0$.

(c) If $k = \alpha$ is a repeated root with multiplicity m of the auxiliary equation for a linear, homogenous differential equation with constant coefficients, then $y = e^{\alpha x}$, $y = xe^{\alpha x}$, $\ldots$, $y = x^{m-1}e^{\alpha x}$ are all solutions.

For more details, start a thread on the discussion board.