EIGENVECTORS AND SEPARATION OF VARIABLES

5 minute review. Remind students how to find eigenvalues using the characteristic equation, and how to find the eigenvectors associated with a given eigenvalue.

Remind students what a differential equation is, the difference between ordinary and partial, linear and non-linear, and what the order of a differential equation is. Go over the principle of separation of variables for solving equations of the form $\frac{dy}{dx} = f(x)g(y)$, using some made-up examples and/or the warm-up below.

Class warm-up. Find the general solution to $\dfrac{dy}{dx} - 2x\sqrt{1-y^2} = 0$.

Problems. (Choose from the below)

I. **Eigenvalues and eigenvectors**. Find the characteristic equation, eigenvectors, and eigenvalues for the following matrices.

$$A = \begin{bmatrix} 1 & 6 \\ 4 & 3 \end{bmatrix}, \qquad F = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

II. **Differential equations for** $\sin$ **and** $\cos$.

(a) Show that, for constants a and b, the function $y = a\cos x + b\sin x$ is a solution to the differential equation

$$\frac{d^2y}{dx^2} + y = 0. \tag{1}$$

(b) Find values for a and b in order to give a solution to Equation (1) satisfying $y = 3$ and $\frac{dy}{dx} = 4$ when $x = 0$.

(c) For which a and b are those solutions to Equation (1) also solutions to the differential equation $\left(\frac{dy}{dx}\right)^2 + y^2 = 1$?

III. (a) Find a solution to

$$\frac{dy}{dx} = e^{y-x}$$

where $y = 1$ at $x = 1$.

(b) Find a solution to

$$x = \ln\left(\frac{dy}{dx}\right) - \ln(y),$$

where $y = e$ at $x = 0$.

IV. **Monomials**. Let m and n be integers. Find the general solution to the differential equation

$$y^n \frac{dy}{dx} = x^m.$$

There are four families of cases to pay attention to:

(a) $m = n = -1$;

(b) $m = -1$, $n \neq -1$;

(c) $m \neq -1$, $n = -1$;

(d) both m and n are different to -1.

For the warm-up, the equations separate to give $\int \frac{1}{\sqrt{1-y^2}}\,dy = \int 2x\,dx$ and the general solution is $y = \sin(x^2 + c)$, for any $c \in \mathbb{R}$.

Selected answers and hints.

I.
- A: $\lambda^2 - 4\lambda - 21 = 0$, eigenvalues $7, -3$ with eigenvectors $[1 \quad 1]^T$ and $[\frac{-3}{2} \quad 1]^T$
- F: $\lambda^3 - 3\lambda - 2 = 0$, eigenvalues $2, -1$ with eigenvector $[1 \quad 1 \quad 1]^T$ associated to 2, and $[1 \quad 0 \quad -1]^T$, $[-1 \quad 1 \quad 0]^T$ associated to -1

II. For part (b) when $x = 0$, $y = a = 3$ and $y' = b = 4$. In part (c) you should find the condition that $a^2 + b^2 = 1$.

III. (a) Write $e^{y-x} = e^y/e^x$ and rearrange. The solution $y = x$ is what's aimed for!

(b) Again, the solution is $y = x$.

(c) Exponentiate both sides and rearrange to get $\frac{1}{y}\frac{dy}{dx} = e^x$; you should find the general solution $y = e^{c+e^x}$, or equivalently $y = ae^{e^x}$. Putting $x = 0$, $y = e$ gives $a = 1$.

IV. Three of the four cases refer to different ways in which a zero could arise in the denominator when you integrate as is. The following are valid forms for the answers in each case:

(a) $y = ax$;

(b) $y^{n+1} = (n+1)\ln|x| + c$;

(c) $y = ae^{x^{m+1}/(m+1)}$;

(d) $y^{n+1} = \frac{n+1}{m+1}x^{m+1} + c$.

For more details, start a thread on the discussion board.

Extra Problems.

I. **Eigenvalues and eigenvectors**. Find the characteristic equation, eigenvectors, and eigenvalues for the following matrices.

$$B = \begin{bmatrix} 5 & -1 \\ 1 & 7 \end{bmatrix}, C = \begin{bmatrix} 5 & 12 \\ 1 & 4 \end{bmatrix},$$

$$D = \begin{bmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{bmatrix}, E = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix},$$

II. **Reversing the problem**. In each case, find a 2×2 matrix with the given information:

(a) eigenvalues 3 and -4,

(b) eigenvalue 2 for eigenvector $[1 \quad -1]^T$,

(c) eigenvalue -3 for eigenvector $[2 \quad 1]^T$ and eigenvalue 6,

(d) eigenvectors $[1 \quad 3]^T$ and $[6 \quad -2]^T$, and

(e) eigenvector $[3 \quad 2]^T$ and eigenvalue 5 for eigenvector $[1 \quad 1]^T$.

III. **A substitution**. Find the general solution to the differential equation

$$(yx^2 + x^3)\frac{dy}{dx} = y^3 + 2y^2x - x^3.$$

(Hint: use the substitution $v = \frac{y}{x}$ and solve for v, and hence for y.)

Selected answers and hints.

I. (a) B: $\lambda^2 - 12\lambda + 36 = 0$, eigenvalue 6 with eigenvector $[1 \quad -1]^T$

(b) C: $\lambda^2 - 9\lambda + 8 = 0$, eigenvalues $1, 8$ with eigenvectors $[-3 \quad 1]^T$ and $[4 \quad 1]^T$

(c) D: $\lambda^3 + \lambda^2 - 12\lambda = 0$, eigenvalues $0, 3, -4$ with eigenvectors $[1 \quad 6 \quad -13]^T$, $[2 \quad 3 \quad -2]^T$ and $[1 \quad -2 \quad -1]^T$

(d) E: $\lambda^3 - 7\lambda^2 + 36 = 0$, eigenvalues $-2, 6, 3$ with eigenvectors $[1 \quad 0 \quad -1]^T$, $[1 \quad 2 \quad 1]^T$ and $[1 \quad -1 \quad 1]^T$

II. Remember that eigenvalues, λ, and eigenvectors, $\begin{bmatrix} x & y \end{bmatrix}^T$ of an arbitrary 2x2 matrix are found from $\begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

You may have other solutions.

(a)

$$\begin{bmatrix} 3 & 0 \\ 0 & -4 \end{bmatrix}$$

(b)

$$\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$$

(c)

$$\begin{bmatrix} 0 & -6 \\ -3 & 3 \end{bmatrix}$$

(d)

$$\begin{bmatrix} \frac{19}{10} & \frac{-3}{10} \\ \frac{-3}{10} & \frac{11}{10} \end{bmatrix}, \quad \text{or even} \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(e)

$$\begin{bmatrix} 2 & 3 \\ -2 & 7 \end{bmatrix}$$

III. Dividing through by x^3, the equation becomes $(\frac{y}{x}+1)\frac{dy}{dx} = (\frac{y}{x})^3 + 2(\frac{y}{x})^2 - 1$. Letting $v = \frac{y}{x}$, it follows that $y = vx$ and hence $\frac{dy}{dx} = v + x\frac{dv}{dx}$. After making the substitution we find that either $v + 1 = 0$ (which gives the solution $y = -x$) or $\frac{dv}{dx} = (v+1)(v-1)/x$, which is separable with general solution $y = x(1 + ax^2)/(1 - ax^2)$ for some a. The v integral requires integration by parts.

For more details, start a thread on the discussion board.