MATRICES: SYSTEMS OF EQUATIONS AND GAUSSIAN ELIMINATION

5 minute review. Remind students about the different types of systems of equations: homogeneous versus non-homogeneous, and singular versus non-singular. Recall how many solutions there are in each case: non-singular = exactly one (which is 0 if homogeneous), singular + non-homogeneous = zero (inconsistent) or infinitely many (use parameters to describe), singular + homogeneous = infinitely many (since 0 is a solution).

Class warm-up. Discuss what kinds of systems each of these are, and therefore the nature of their solution sets. Find the solutions.

$$\text{(a)}\ \begin{aligned} 2x - 5y &= 1 \\ x + 3y &= 6 \end{aligned} \qquad \text{(b)}\ \begin{aligned} x_1 - x_2 + x_3 &= 0 \\ 3x_1 + x_2 - 4x_3 &= 0 \\ 5x_1 - x_2 - 2x_3 &= 0 \end{aligned}$$

Problems. (Choose from the below)

I. **Standard systems of equations**. Solve the following systems of equations, using matrix methods where possible.

$$\text{(a)}\ \begin{aligned} 2x_1 + 7x_3 &= 0 \\ x_1 - x_2 + 3x_3 &= 0 \\ 4x_1 + 5x_2 + 2x_3 &= 0 \end{aligned} \qquad \text{(b)}\ \begin{aligned} 2x_1 + 3x_3 &= 1 \\ x_1 - x_2 + 5x_3 &= 1 \\ 2x_2 - 4x_3 &= -2 \end{aligned}$$

II. **Singular systems**. Find the value(s) of α for which the following systems of equations have infinitely many solutions and then find those solutions, using matrix methods where possible.

$$\text{(a)}\ \begin{aligned} -x_1 + 4x_2 + \alpha x_3 &= -2 \\ 2x_1 - x_2 &= -1 \\ 5x_1 + x_2 - x_3 &= -5 \end{aligned} \qquad \text{(b)}\ \begin{aligned} 6x_1 + 3x_2 - 7x_3 &= 1 \\ x_1 + 4x_2 + x_3 &= -1 \\ 9x_1 + 15x_2 - 4x_3 &= \alpha \end{aligned}$$

III. **Gaussian elimination versus matrix methods: systems of equations**. For each of the systems below, solve using both Gaussian elimination and matrix methods. Which do you prefer and why? Perhaps race your neighbour and compare results.

(a)

$$\begin{aligned} 4x_2 + 2x_3 &= 1 \\ x_1 - x_3 &= 0 \\ -3x_1 + 8x_2 + 3x_3 &= 5 \end{aligned}$$

(b)

$$\begin{aligned} 4x_1 - x_2 + 5x_3 &= 10 \\ 3x_1 + 2x_2 + 6x_3 &= -1 \\ -x_1 + 4x_2 &= 0; \end{aligned}$$

IV. **Gaussian elimination versus matrix methods: inverses**. For each matrix, find the inverse using both Gaussian elimination and matrix methods. Once again, which do you prefer and why?

$$A = \begin{bmatrix} 1 & 6 & 0 \\ 3 & 3 & -3 \\ 0 & 7 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & 6 & -2 & 1 \\ 3 & 0 & 0 & 4 \\ 1 & 1 & 1 & -1 \\ 2 & 0 & -2 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 3 & 1 & 5 \\ 1 & 0 & 3 & 1 \\ 0 & 2 & -3 & 2 \\ 0 & 2 & 3 & 1 \end{bmatrix}.$$

For the warm-up, (a) is non-homogeneous and $|A| = 11$, so is non-singular. $A^{-1} = \frac{1}{11}\begin{bmatrix} 3 & 5 \\ -1 & 2 \end{bmatrix}$, therefore the solution is $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

For (b), $|A| = 0$, and so it's singular. The equations reduce to:

$$x_1 - x_2 + x_3 = 0, \quad 4x_1 - 3x_3 = 0$$

Hence the solution is $\mathbf{x} = [\frac{3}{4}\lambda \quad \frac{7}{4}\lambda \quad \lambda]$.

Selected answers and hints.

I. (a) $|A| = 29 \neq 0$ so non-singular and hence the only solution is $[0 \quad 0 \quad 0]^T$.

(b) $|A| = -6$ hence non-singular so an inverse can be obtained. $A^{-1} = -\frac{1}{6}\begin{bmatrix} -6 & 6 & 3 \\ 4 & -8 & -7 \\ 2 & -4 & -2 \end{bmatrix}$.

Hence the solution is $[1 \quad \frac{-5}{3} \quad \frac{-1}{3}]^T$.

II. In cases (a) we can find α using $|A| = 0$; the fact that $|A| = 0$ also means that A has no inverse and so we solve the equations directly.

(a) $\alpha = -1$; the solutions are $x_1 = \lambda, x_2 = 2\lambda + 1, x_3 = 7\lambda + 6$.

(b) $\alpha = -2$ is necessary to make the equations consistent (e.g. Eq 1 + 3*Eq 2 = Eq 3); the solutions are $x_1 = \lambda, x_2 = \frac{-13\lambda - 6}{31}, x_3 = \frac{21\lambda - 7}{31}$

III. (a) $A^{-1} = \frac{1}{16}\begin{bmatrix} 8 & 4 & -4 \\ 0 & 6 & 2 \\ 8 & -12 & -4 \end{bmatrix}$. The solution is $[\frac{-3}{4} \quad \frac{5}{8} \quad \frac{-3}{4}]^T$.

(b) $A^{-1} = -\frac{1}{20}\begin{bmatrix} -24 & 20 & -16 \\ -6 & 5 & -9 \\ 14 & -15 & 11 \end{bmatrix}$. The solution is $[13 \quad \frac{13}{4} \quad \frac{-31}{4}]^T$.

IV. (a)

$$A^{-1} = \frac{1}{6}\begin{bmatrix} 24 & -6 & -18 \\ -3 & 1 & 3 \\ 21 & -7 & -15 \end{bmatrix}$$

(b)

$$B^{-1} = \frac{1}{42}\begin{bmatrix} -8 & -10 & 48 & 32 \\ 13 & 11 & -36 & -31 \\ 1 & 17 & -6 & -25 \\ 6 & 18 & -36 & -24 \end{bmatrix}$$

(c)

$$C^{-1} = \begin{bmatrix} 18 & -35 & -28 & 1 \\ 9 & -18 & -14 & 1 \\ -2 & 4 & 3 & 0 \\ -12 & 24 & 19 & -1 \end{bmatrix}$$

For more details, start a thread on the discussion board.