MORE DETERMINANTS AND INVERSES

5 minute review. Remind students how to compute determinants for any $n \times n$ matrix using any row/column.

Remind students how to compute the cofactor $C_{ij} = (-1)^{i+j} \det(M_{ij})$, where M_{ij} is the matrix obtained from A by deleting row i and column j. The adjoint of A is then defined to be the transpose of the matrix of cofactors, $\operatorname{adj}(A) = C^T$, and the inverse of A (assuming $\det(A) \neq 0$) is given by $A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)$. Show how this gives formula for 2×2 matrices.

Class warm-up. Find the matrix of cofactors, the determinant, and the inverse for the matrix

$$A = \begin{bmatrix} 1 & 6 & 4 \\ 9 & -1 & 2 \\ 0 & 3 & 3 \end{bmatrix}.$$

Problems. (Choose from the below)

I. **Determinants using row/column operations.**. Use row and column operations to help you compute the determinant of the following 4×4 matrix:

$$A = \begin{bmatrix} 3 & 15 & -4 & 2 \\ 1 & 1 & 1 & 7 \\ 5 & -3 & -5 & 6 \\ -5 & 0 & 1 & 3 \end{bmatrix}$$

II. **Inverses.**. Find the inverses of the following matrices:

$$A = \begin{bmatrix} 3 & 1 \\ -2 & 7 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 3 \\ -8 & -6 \end{bmatrix}. \quad C = \begin{bmatrix} 1 & 0 & 6 \\ -2 & 2 & 1 \\ -3 & 5 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 3 & 1 & 1 \\ -1 & 4 & 4 & 4 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

III. **Are inverses unique?**. Can a matrix have two different inverses? Let A be a matrix and let B and C both be inverses. What does this mean and does $B = C$? (Hint: use the definition, not the formula.)

IV. **Solving matrix equations**. Solve for the matrix B_i in each equation below. Note the size and shape!

$$\begin{bmatrix} 7 & 2 & 1 \\ 0 & 3 & -1 \\ -3 & 4 & -2 \end{bmatrix} B_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix} B_2 = \begin{bmatrix} 2 & 4 & 5 \\ 1 & -1 & 0 \\ 6 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} B_3 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ 0 & 5 & 2 & 0 \\ 6 & 1 & 8 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 8 & 0 \\ -3 & 2 & 0 & 4 \\ -2 & -4 & -8 & 0 \end{bmatrix}$$

For the warm-up, $A^{-1} = \frac{1}{-63}\begin{bmatrix} -9 & -6 & 16 \\ -27 & 3 & 34 \\ 27 & -3 & -55 \end{bmatrix}$.

Selected answers and hints.

I. $|A| = 3623$

II.

$$A^{-1} = \frac{1}{23}\begin{bmatrix} 7 & -1 \\ 2 & 3 \end{bmatrix}, \quad C^{-1} = \frac{1}{-29}\begin{bmatrix} -5 & 30 & -12 \\ -3 & 18 & -13 \\ -4 & -5 & 2 \end{bmatrix}, \quad D^{-1} = \frac{1}{-22}\begin{bmatrix} -16 & -8 & 6 & 16 \\ 2 & -10 & 2 & -2 \\ -3 & 4 & -3 & -8 \\ -3 & 4 & -3 & 14 \end{bmatrix}$$

B^{-1} does not exist.

III. Inverses are unique; if B, C are both inverses for A then

$$AB = I, \quad BA = I, \quad AC = I, \quad CA = I,$$

so

$$B = BI = BAC = IC = C.$$

IV. For any equation $AB = C$, to solve for B we multiply by A^{-1} to get $B = A^{-1}C$.

$$B_1 = \begin{bmatrix} -12 \\ 17 \\ 51 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & -114 & -120 \\ 1 & 95 & 100 \\ 0 & -24 & -25 \end{bmatrix}, \quad B_3 = \begin{bmatrix} \frac{51}{2} & -8 & 4 & 0 \\ -49 & 12 & -4 & 2 \\ -13 & 4 & -2 & 0 \end{bmatrix}$$

For more details, start a thread on the discussion board.

Extra Problems.

I. **Determinants using row/column operations.**. Use row and column operations to help you compute the determinants of the following 4×4 matrices:

$$B = \begin{bmatrix} 2 & 1 & 0 & 4 \\ 7 & 3 & -7 & 10 \\ -2 & 5 & 6 & -11 \\ -1 & 8 & 15 & 8 \end{bmatrix}, C = \begin{bmatrix} 3 & 2 & 1 & -3 \\ 5 & 3 & 0 & 9 \\ 4 & 8 & 2 & 5 \\ -7 & 11 & 3 & -7 \end{bmatrix}.$$

II. **A large determinant.**. Use your method of choice to compute the determinant of

$$A = \begin{bmatrix} -12 & 1 & -8 & 16 & 5 & 6 \\ 5 & 8 & 3 & 9 & 1 & -27 \\ 21 & -1 & 2 & 15 & -3 & 24 \\ -7 & 2 & 10 & 1 & 9 & 11 \\ 2 & 3 & 7 & 8 & 4 & 6 \\ 21 & 3 & 6 & 1 & 2 & 10 \end{bmatrix}.$$

III. **Inverses.**. Find the inverses of the following matrices.

$$B = \begin{bmatrix} 2 & -4 \\ -4 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 1 & 1 \\ -1 & 3 & -5 \\ 11 & 9 & 1 \end{bmatrix},$$

IV. **Transpose and inverse**. Here we investigate $(A^T)^{-1}$ and $(A^{-1})^T$.

(a) Start with a 2×2 matrix of your choice, find its inverse, and then take the transpose; now start with the same matrix, take the transpose, and find the inverse of that matrix. Now try it with a simple 3×3 matrix.

(b) Find the matrix of cofactors for A^T. How does it relate to the matrix of cofactors for A?

(c) Use the formula for the inverse with cofactors and determinants to show that $(A^T)^{-1} = (A^{-1})^T$.

(d) Recall that a symmetric matrix A is one that satisfies $A = A^T$. An example is

$$A = \begin{bmatrix} 1 & 0 & 6 \\ 0 & 3 & 2 \\ 6 & 2 & -2 \end{bmatrix}.$$

Find the inverse of A, and check that A^{-1} is also symmetric. Can you show that if a symmetric matrix A is invertible, then A^{-1} is also symmetric using the cofactor formula for the inverse?

(e) A skew symmetric matrix B is one that satisfies $-B = B^T$. An example is

$$B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Find the inverse of B, and check that B^{-1} is also skew symmetric. Now try a 3×3 skew symmetric matrix and see what happens.

Selected answers and hints.

I. $|B| = 1196, |C| = -265$

II. $|A| = -840596 = -4 \cdot 31 \cdot 6779$

III.
$$B^{-1} = \frac{1}{-14}\begin{bmatrix} 1 & 4 \\ 4 & 2 \end{bmatrix}$$

C^{-1} does not exist since $|C| = 0$.

IV. (a) We will see later that $(A^T)^{-1} = (A^{-1})^T$.

(b) The cofactor A_{ij} is the same as the (j, i) cofactor of A^T: $A_{ij} = (A^T)_{ji}$.

(c) This follows from the previous part plus the fact that $|A| = |A^T|$.

(d) For the matrix given, $A^{-1} = \frac{1}{-118}\begin{bmatrix} -10 & 12 & -18 \\ 12 & -38 & -2 \\ -18 & -2 & 3 \end{bmatrix}$.

In general, if A is invertible, then $(A^{-1})^T = (A^T)^{-1} = A^{-1}$ so A^{-1} is symmetric.

(e) $B^{-1} = -B$ so is still skew symmetric. A 3×3 skew symmetric matrix B is never invertible since $|B| = 0$, so it doesn't make sense to see if B^{-1} is also skew symmetric.

For more details, start a thread on the discussion board.