MATRICES AND DETERMINANTS

5 minute review. Remind students how to multiply matrices (including that in AB, A must be $m \times n$, B must be $n \times p$, and the result is $m \times p$). Remind students how to compute determinants (both 2×2 and 3×3). In the 3×3 case, explain that you can use different rows or columns. As an example, you could show that $\begin{vmatrix} 4 & 13 & 4 \\ 0 & -1 & 1 \\ 3 & 7 & 5 \end{vmatrix} = 3$.

Class warm-up. Explain how a 2×2 matrix A gives a function $\mathbb{R}^2 \to \mathbb{R}^2$ using matrix multiplication. "Draw" the functions associated with the matrices below by seeing what each matrix does to a picture in a square of side-length 2. (Any picture you can draw with clear top, bottom, left, and right sides will work; a stick-person style unicorn, perhaps. Then map the square using the matrix, and fill in the picture afterwards, taking care with orientation.)

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \quad A_4 = \begin{bmatrix} -1 & 3 \\ 1 & 1 \end{bmatrix}.$$

Use the matrices above to investigate how the sign of the determinant of a 2×2 matrix relates to whether it preserves orientation or not as a map $\mathbb{R}^2 \to \mathbb{R}^2$.

Problems. (Choose from the below)

I. **Symmetric and skew-symmetric matrices**. A matrix A is called *symmetric* if $A^T = A$ and *skew-symmetric* if $A^T = -A$. A general 2×2 matrix is of the form $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ for real numbers a, b, c, d.

  (a) Write down a general 2×2 symmetric matrix.

  (b) Write down a general 2×2 skew-symmetric matrix.

  (c) Can you express $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ as the sum of a symmetric and a skew-symmetric matrix? What about a general matrix A? What about the 3×3 case?

II. **Orthogonal matrices**. A square matrix is *orthogonal* if $AA^T = I$.

  (a) Show that $A_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is orthogonal for any angle θ.

  (b) Find four different 2×2 orthogonal matrices with *integer* entries.

  (c) Is every 2×2 orthogonal matrix of the form A_θ for some angle θ?

III. **A formula for** $|AB|$.

  (a) Let $A = \begin{bmatrix} 3 & 1 \\ -2 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -4 \\ -4 & 1 \end{bmatrix}$. Show that $|AB| = |A| \times |B|$ holds in this case.

  (b) Now let $A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$ and $B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$. Show that $|AB| = |A| \times |B|$ holds for the general 2×2 case.

  (c) Can you do the general 3×3 case?

IV. 3×3 **determinants**. Find the determinants of the following 3×3 matrices by expanding along a row or column of your choice:

$$A = \begin{bmatrix} 5 & 1 & 0 \\ -6 & 0 & 3 \\ 1 & 1 & 7 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -2 & 9 \\ 1 & 1 & 4 \\ -3 & 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 & 5 \\ -2 & 1 & 2 \\ 1 & 7 & 17 \end{bmatrix}, \quad D = \begin{bmatrix} 8 & -3 & -1 \\ 5 & 7 & 2 \\ -2 & 4 & 1 \end{bmatrix}.$$

Justify why you chose the row or column you did in each case.

For the warm-up,

- A_1 is the identity function, it doesn't change the square at all.
- A_2 swaps the x and y coordinates; that is, it flips the square across the line $y = x$.
- A_3 sends (x, y) to $(x + 2y, y)$, so turns the square into a parallelogram.
- A_4 sends (x, y) to $(3y - x, x + y)$.

Selected answers and hints.

I. (a)
$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

(b)
$$A = \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}$$

(c) For the 2×2 case, $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & \frac{1}{2}(b+c) \\ \frac{1}{2}(b+c) & d \end{bmatrix} + \begin{bmatrix} 0 & \frac{1}{2}(b-c) \\ \frac{1}{2}(c-b) & 0 \end{bmatrix}$. This can be written as $A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$, which also works for bigger matrices.

II. (a) $AA^T = I$ follows from $\sin^2\theta + \cos^2\theta = 1$.

(b) The possibilities are $\begin{bmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{bmatrix}$.

(c) Not quite. Every orthogonal matrix is either of the form $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, which represents a rotation as a function $\mathbb{R}^2 \to \mathbb{R}^2$, or $\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$, which represents a reflection.

III. $|A| = 23, |B| = -14, |AB| = -322$

IV. (a) $|A| = 30$ is best computed using column 2.

(b) $|B| = 51$ is best computed using row 3.

(c) $|C| = 0$ is best computed using column 1 (fewest negative signs).

(d) $|D| = -15$ with no clear choice of best row or column.

For more details, start a thread on the discussion board.

Extra Problems.

I. **Triangular matrices**. An upper triangular 3×3 matrix is one of the form $\begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$.

(a) Express $A = \begin{bmatrix} 1 & 6 & -1 \\ 2 & 0 & 3 \\ 4 & -4 & 4 \end{bmatrix}$ as the sum of an upper triangular matrix and a symmetric matrix. Can you do this in more than one way?

(b) Now find a matrix which is both upper triangular and symmetric, but is not the zero matrix.

(c) Now express the matrix A as the sum of an upper triangular and *skew-symmetric* matrix. Can you do this in more than one way?

(d) Can you find a matrix which is both upper triangular and skew-symmetric?

II. **Linear combinations***.

(a) Can you obtain the identity matrix from the matrices

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 1 \\ 4 & -1 \end{bmatrix}$$

using only the operations of matrix addition, scalar multiplication, and transpose? In other words, can you find scalars $\alpha, \beta, \gamma, \delta$ such that

$$\alpha A + \beta A^T + \gamma B + \delta B^T = I_2?$$

(b) Can you choose different matrices A, B above that will solve this equation? If you choose A, B "at random," do you think you can solve this equation or not?

III. **Matrices and functions**. Let A, B, C, D be the matrices below.

$$A = \begin{bmatrix} 1 & -2 \\ 3 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 4 \\ -1 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 7 & 0 & 1 \\ 1 & 2 & 5 \end{bmatrix}, \quad D = \begin{bmatrix} -3 & 1 \\ -1 & 0 \\ 2 & 4 \end{bmatrix}.$$

Compute AB, AC, BB^T, B^TB, CD and DC.

IV. **Cancellation of matrices**.

(a) Find two distinct 3×3 matrices A, B such that $AC = BC$ where C is

$$C = \begin{bmatrix} 1 & 3 & -4 \\ 2 & 1 & 7 \\ 1 & -2 & 11 \end{bmatrix}.$$

(Hint: $AC = BC$ is equivalent to $(A - B)C = 0$.)

(b) Can you pick A, B such that both $AC = BC$ and $CA = CB$ hold?

V. **The characteristic polynomial**. Let

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 4 \\ 3 & -2 \end{bmatrix}.$$

(a) Find A^2 and A^3.

(b) Check that $A^3 - 2A^2 - A = 0$.

(c) Now find B^2 and B^3.

(d) Can you find integers r, s such that $B^3 + rB^2 + sB = 0$?

VI. **Permutation matrices**. A permutation matrix is an $n \times n$ matrix P such that each row and each column has a single 1 and the rest 0's in it. As an example, the identity matrix I_n is a permutation matrix for any choice of n.

(a) There are exactly two permutation matrices of size 2×2, the identity and one other. Find the other permutation matrix, and its determinant.

(b) There are exactly six permutation matrices of size 3×3. Find them, and compute all the determinants.

(c) *Do you notice a pattern in the determinants of permutation matrices?

(d) *Not all matrices whose entries are only 0 or 1 have determinant ± 1. Find a 3×3 matrix A whose entries are only 0 or 1 such that $|A| = 0$. Can you find a 3×3 matrix B whose entries are only 0 or 1 such that $|B|$ is not $0, 1$ or -1?

Selected answers and hints.

I. (a) Many possible answers. One is

$$\begin{bmatrix}1 & 6 & -1\\ 2 & 0 & 3\\ 4 & -4 & 4\end{bmatrix} = \begin{bmatrix}1 & 2 & 4\\ 2 & 0 & -4\\ 4 & -4 & 4\end{bmatrix} + \begin{bmatrix}0 & 4 & -5\\ 0 & 0 & 7\\ 0 & 0 & 0\end{bmatrix}.$$

(b) The identity matrix, or any matrix with nonzero entries only along the main diagonal, is symmetric and upper triangular.

(c) Only one possible solution,

$$\begin{bmatrix}1 & 6 & -1\\ 2 & 0 & 3\\ 4 & -4 & 4\end{bmatrix} = \begin{bmatrix}0 & -2 & -4\\ 2 & 0 & 4\\ 4 & -4 & 0\end{bmatrix} + \begin{bmatrix}1 & 8 & 3\\ 0 & 0 & -1\\ 0 & 0 & 4\end{bmatrix}.$$

(d) The only matrix which is upper triangular and skew-symmetric is the zero matrix. If A is upper triangular, then it has zeroes below the diagonal. If it is skew-symmetric it has them along the diagonal as well, and then $A^T = -A$ means it has them above the diagonal.

II. (a) Not possible with the A, B given.

(b) Random choices will not work. Does work with

$$A = \begin{bmatrix}1 & 2\\ 3 & 1\end{bmatrix}, \quad B = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$$

as $I = A - 2B - 3B^T$.

III.

$$AB = \begin{bmatrix}6 & 2\\ 12 & 12\end{bmatrix}, \quad AC = \begin{bmatrix}5 & -4 & -9\\ 21 & 0 & 3\end{bmatrix}, \quad BB^T = \begin{bmatrix}32 & 0\\ 0 & 2\end{bmatrix},$$

$$B^TB = \begin{bmatrix}17 & 15\\ 15 & 17\end{bmatrix}, \quad CD = \begin{bmatrix}-19 & 11\\ 5 & 21\end{bmatrix}, \quad DC = \begin{bmatrix}-20 & 2 & 2\\ -7 & 0 & -1\\ 18 & 8 & 22\end{bmatrix}.$$

IV. (a) Pick any A, B such that

$$A - B = \begin{bmatrix}\alpha & -\alpha & \alpha\\ \beta & -\beta & \beta\\ \gamma & -\gamma & \gamma\end{bmatrix}$$

for any constants α, β, γ.

(b) Any A, B for which $A - B$ is a scalar multiple of $\begin{bmatrix}-5 & 5 & -5\\ 3 & -3 & 3\\ 1 & -1 & 1\end{bmatrix}$ works.

V. (a)

$$A^2 = \begin{bmatrix}3 & 2\\ 4 & 3\end{bmatrix}$$

$$A^3 = \begin{bmatrix}7 & 5\\ 10 & 7\end{bmatrix}$$

(b) Easy to verify.

(c)

$$B^2 = \begin{bmatrix}13 & -4\\ -3 & 16\end{bmatrix}$$

$$B^3 = \begin{bmatrix}1 & 60\\ 45 & -44\end{bmatrix}$$

(d) $B^3 + B^2 - 14B = 0$

VI. (a) The other permutation matrix is

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and its determinant is -1.

(b) Of the six 3×3 permutation matrices, half have determinant 1 and half have determinant -1.

(c) There are $n! = n(n-1)(n-2)\cdots 2 \cdot 1$ permutation matrices of size $n \times n$, half have determinant 1 and half have determinant -1.

(d)

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

have determinants $|A| = 0$ and $|B| = -2$.

For more details, start a thread on the discussion board.