

## IMPROPER INTEGRALS

**5 minute review.** Recap two types of improper integral: (1) integrals over an infinite domain, and (2) integrals with isolated points where the integrand is not defined. Explain how both may be defined using limits:

$$\int_a^\infty f(x)dx = \lim_{X \rightarrow \infty} \int_a^X f(x)dx, \quad \int_a^b g(x)dx = \lim_{\epsilon \rightarrow 0^+} \int_a^{c-\epsilon} g(x)dx + \lim_{\epsilon \rightarrow 0^+} \int_{c+\epsilon}^b g(x)dx.$$

(Here assume  $g(x)$  is not defined at  $x = c$ .) Also, use  $[F(x)]_a^\infty = \lim_{c \rightarrow \infty} [F(x)]_a^c$ , etc.

**Class warm-up.** With class input, decide which of the following are well-defined and finite, and evaluate where possible.

$$\int_0^\infty x dx, \quad \int_0^\infty x^{-2} dx, \quad \int_1^\infty \ln x dx, \quad \int_0^1 \ln x dx, \quad \int_{-1}^2 |x|^3 dx.$$

**Problems.** (Choose from the below)

I. Which of these integrals are valid? Evaluate those that are well-defined.

- |                                    |                                      |
|------------------------------------|--------------------------------------|
| (a) $\int_1^\infty \frac{1}{x} dx$ | (f) $\int_0^\infty \sin(x)e^{-x} dx$ |
| (b) $\int_0^1 \frac{1}{x} dx$      | (g) $\int_{-\infty}^0 e^x dx$        |
| (c) $\int_1^\infty x^{-1.1} dx$    | (h) $\int_0^{\pi/2} \tan x dx$       |
| (d) $\int_0^1 x^{-0.9} dx$         | (i) $\int_{-1}^1  x  dx$             |
| (e) $\int_0^1 x \ln(x) dx$         | (j) $\int_{-1}^2 \frac{1}{x^3} dx$   |

II. **The Gamma function.** The Gamma function  $\Gamma(n)$  is defined by an improper integral:  $\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx$ .

- (a) Show that  $\Gamma(1) = 1$ .
- (b) By integrating by parts, show that  $\Gamma(n) = (n-1)\Gamma(n-1)$  for  $n > 1$ . Hence show that  $\Gamma(n) = (n-1)!$  when  $n$  is any positive integer.
- (c) Now let  $n$  be a real number. For which values of  $n$  is  $\Gamma(n)$  well-defined according to this definition?

III. **The area under a Gaussian\***. The ‘Gaussian’ function  $f(x) = \exp(-x^2)$  appears in many contexts; for example, in the ‘normal distribution’ (‘bell curve’) in probability, which arises as the large- $N$  limit of the binomial distribution. Here we consider the total area under a Gaussian,

$$I = \int_{-\infty}^\infty \exp(-x^2) dx.$$

- (a) Sketch the integrand. Is  $I$  well-defined?
- (b) The integral can be found using the following result:

$$I^2 = \int_0^\infty 2\pi r \exp(-r^2) dr$$

(ask your tutor where this comes from!). Use the formula above to find  $I^2$  and hence  $I$ . Use your new result to show that  $\Gamma(1/2) = \sqrt{\pi}$ .

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For the warm-up, the answers are (in order): undefined [upper limit]; undefined [lower];  
undefined [upper],  $-1$ ,  $17/4$ .

**Selected answers and hints.**

I.

- |                                  |  |
|----------------------------------|--|
| (a) divergent due to upper limit | (f) $1/2$  |
| (b) divergent due to lower limit | (g) $1$  |
| (c) $10$                         | (h) divergent due to upper limit   |
| (d) $10$                         | (i) $1$  |
| (e) $-1/4$                       | (j) ill-defined (can't find the limit as $x \rightarrow 0$ ).<br>Could argue by symmetry for $3/8$ . |

II. (c) For  $n > 0$ .

III. (a) Yes. (b)  $I = \sqrt{\pi}$ . To find  $\Gamma(1/2)$ , let  $u = x^2$ , and show that  $I = 2 \int_0^\infty e^{-x^2} dx = \int_0^\infty u^{-1/2} e^{-u} du$ . Now compare with the definition of  $\Gamma(n)$ .

For more details, start a thread on the discussion board.