FURTHER INTEGRATION

5 minute review. Remind students that hyperbolic substitutions can solve integrals which trigonometric substitutions can't, such as $\int \frac{dx}{\sqrt{1+x^2}}$ ($x = \sinh u$), $\int \frac{dx}{\sqrt{x^2-1}}$ ($x = \cosh u$) and $\int \frac{dx}{1-x^2}$ ($x = \tanh u$). Also remind students that there is a trick to integrating rational functions of $\sin x$ and $\cos x$ using the substitution $t = \tan(\frac{x}{2})$, which gives expressions for $\sin x$ and $\cos x$ in terms of t; see QII.

Briefly recap $\int_a^b f(x)dx = F(b) - F(a)$, where $\frac{dF}{dx} = f$.

Class warm-up. Find $\int \sqrt{1+x^2}\, dx$. If more examples are desired, choose something from the below.

Integrate $\int_0^1 xe^{-x}dx$, using integration by parts and $[\,]_a^b$ notation.

Problems. (Choose from the below)

I. **Find the following indefinite integrals:**.

(a) $\displaystyle\int \sqrt{x^2-9}\, dx$

(b) $\displaystyle\int \sqrt{x^2+4x-5}\, dx$

(c) $\displaystyle\int \left(1-x^2\right)^{3/2} dx$

(d) $\displaystyle\int \frac{\sinh x}{\sinh x + \cosh x}\, dx$

II. **The $t = \tan(x/2)$ substitution**. Consider $I = \int \frac{dx}{\sin x + \cos x}$.

(a) Use a double-angle formula to show that $\sin(x + \pi/4) = \frac{1}{\sqrt{2}}(\sin x + \cos x)$. Hence use the substitution $y = x + \pi/4$ to write $I = \frac{1}{\sqrt{2}} \int \frac{dy}{\sin y}$.

(b) Let $t = \tan(y/2)$. Show that (i) $\frac{dy}{dt} = \frac{2}{1+t^2}$ and (ii) $\sin y = \frac{2t}{1+t^2}$.

(c) Show that $I = \frac{1}{\sqrt{2}} \ln\left|\tan\left(\frac{x}{2} + \frac{\pi}{8}\right)\right| + c$.

III. **Definite integrals**. Evaluate the following definite integrals:

(a) $\displaystyle\int_1^2 \ln x\, dx$

(b) $\displaystyle\int_1^4 \frac{dt}{(1+t)^2}$

(c) $\displaystyle\int_1^{\sqrt{3}} \frac{dx}{1+x^2}$

(d) $\displaystyle\int_0^{\ln 2} \sqrt{e^x - 1}\, dx$ (Hint: let $u = \sqrt{e^x-1}$.)

IV. **Integration from first principles***. Let $I = \int_0^a e^x dx$. In this question we will show that $I = e^a - 1$ by applying 'integration from first principles', using

$$\int f(x)dx = \lim_{n\to\infty} \sum_{k=0}^{n-1} f(x_k)h_k, \quad \text{where} \quad h_k = x_{k+1} - x_k.$$

(a) Sketch the area represented by the integral. Now split the area into n equal-width bars. Show that: (i) the width of each bar is $h_k = h = a/n$, and (ii) $x_k = kh$. By inserting $f(x_k) = e^{kh}$ in the first principles formula, show that

$$I = \lim_{n\to\infty} \frac{a}{n} S_n, \qquad \text{where} \quad S_n = \sum_{k=0}^{n-1} e^{ka/n}$$

(b) Show that S_n is a geometric series, $S_n = 1 + b + b^2 + \ldots + b^{n-1}$, where $b = e^{a/n}$. Sum the geometric series to show that $S_n = \frac{e^a - 1}{e^{a/n} - 1}$.

(c) By using the Maclaurin series expansion for $e^{a/n}$ and taking the limit $n \to \infty$, show that $I = e^a - 1$ as expected.

For the warm-up, $\int \sqrt{1+x^2}\,dx = \frac{1}{2}x\sqrt{1+x^2} + \frac{1}{2}\sinh^{-1} x$ using the substitution $x = \sinh u$ and the identity $\cosh^2 u = \frac{1}{2}(1+\cosh(2u))$.

$$\int_0^1 xe^{-x}dx = \left[-xe^{-x}\right]_0^1 + \int_0^1 e^{-x}dx = \left[-(x+1)e^{-x}\right]_0^1 = -2e^{-1} + 1\times e^0 = 1-2e^{-1} \approx 0.264.$$

Selected answers and hints. (All answers should include a constant of integration.)

I. (a) $\frac{1}{2}x\sqrt{x^2-9} - \frac{9}{2}\cosh^{-1}(x/3)$
(b) $\frac{1}{2}(x+2)\sqrt{x^2+4x-5} - \frac{9}{2}\cosh^{-1}\left(\frac{1}{3}(x+2)\right)$
(c) $\frac{1}{4}x\sqrt{1-x^2}\left(\frac{5}{2}-x^2\right) + \frac{3}{8}\sin^{-1} x$
(d) $\frac{1}{2}x + \frac{1}{4}e^{-2x}$

II. (b)(i) $\frac{dt}{dy} = \frac{1}{2}\sec^2(y/2) = \frac{1}{2}\left(1+\tan^2(y/2)\right) = \frac{1}{2}\left(1+t^2\right)$ so $\frac{dy}{dt} = \frac{2}{1+t^2}$.
(b)(ii) $\sin(y) = 2\sin(y/2)\cos(y/2) = 2\tan(y/2)/\sec^2(y/2) = 2t/(1+t^2)$.

III. (a) $2\ln 2 - 1$, (b) $3/10$, (c) $\pi/12$, (d) $2-\pi/2$

For more details, start a thread on the discussion board.

Extra Problems.

I. **Find the following indefinite integrals:**.

(a) $\displaystyle\int \sqrt{25-x^2}\,dx$ (b) $\displaystyle\int \frac{\sin x}{\sin x+\cos x}\,dx$

II. **Integrals of $\sqrt{a^2-x^2}$**. Let $I=\int\sqrt{a^2-x^2}\,dx$ in the region $|x|<|a|$.

(a) By making the substitution $x=a\sin u$, show that

$$I=\frac{1}{2}x\sqrt{a^2-x^2}+\frac{1}{2}a^2\sin^{-1}\left(\frac{x}{a}\right)+c.$$

(b) Here's another approach. Starting with $I=\int\sqrt{a^2-x^2}\,dx$ and treating it as a function of both x and a, one can show that $\frac{\partial I}{\partial a}=a\sin^{-1}(x/a)$ (how?). Now, integrate with respect to a, treating x as a constant and using integration by parts with $u=\sin^{-1}(x/a)$ and $\frac{dv}{da}=a$.

III. **Consistency of integrals**. Some integrals can be expressed in more than one way, and the results can look superficially rather different. For example, $\int\frac{dx}{1-x^2}$ can be written as both $\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)+c$ and $\tanh^{-1}(x)+c$.

(a) By starting with $\tanh(y)=\frac{e^y-e^{-y}}{e^y+e^{-y}}=x$, show that $\tanh^{-1}x=y=\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$.

(b) Show that the following expressions are consistent, and find the relationships between c_1 and c_2.

(i) $\int\frac{dx}{\sqrt{1-x^2}}=\sin^{-1}x+c_1=-\cos^{-1}x+c_2$ (where $-1<x<1$).

(ii) $\int\frac{dx}{\sqrt{x^2-1}}=\cosh^{-1}x+c_1=\ln\left(x+\sqrt{x^2-1}\right)+c_2$ (where $x\geq 1$).

IV. **Definite integrals**. Evaluate the following definite integrals:

(a) $\displaystyle\int_{1/2}^{3/4}\frac{dt}{(1-t)(1+2t)}$ (b) $\displaystyle\int_0^{\pi}e^{-x}\sin(x)dx$

V. **Definite integrals with functions of x in their limits**.

(a) Let $I(x)=\int_x^{x^2}y\ln(y)dy$. By integrating, show that $I(x)=\frac{x^2}{4}(1-x^2)+\frac{x^2}{2}(2x^2-1)\ln|x|$. Now by differentiating your answer, show that $\frac{dI}{dx}=(4x^2-1)x\ln|x|$.

(b) Apply the following general formula to reach the same result:

$$\frac{d}{dx}\int_{a(x)}^{b(x)}f(y)dy=\frac{db}{dx}\times f(b(x))-\frac{da}{dx}\times f(a(x)).$$

(c) Consider how the region of integration changes when x changes slightly, $x\to x+\epsilon$. Recall that $a(x+\epsilon)\approx a(x)+\epsilon a'(x)$. Use differentiation from first principles and the approach of Q2 to derive the formula above.

Selected answers and hints. (All answers should include a constant of integration.)

I. (a) $\frac{1}{2}x\sqrt{25-x^2}+\frac{25}{2}\sin^{-1}(x/5)$
(b) $\frac{x}{2}-\frac{1}{2}\ln|\sin(x)+\cos(x)|$

II. Here's one justification of the given expression for $\frac{\partial I}{\partial a}$. Given that $I=\int\sqrt{a^2-x^2}\,dx$, by definition that means that the derivative of I with respect to x is $\sqrt{a^2-x^2}$. In other words, if we treat I as a function of both x and a, then $\frac{\partial I}{\partial x}=\sqrt{a^2-x^2}$. Now, differentiate partially with respect to a to get $\frac{\partial^2 I}{\partial a\partial x}=\frac{a}{\sqrt{a^2-x^2}}$. Since $\frac{\partial^2 I}{\partial a\partial x}$ will also be the result of differentiating $\frac{\partial I}{\partial a}$ with respect to x keeping a constant, it follows that $\frac{\partial I}{\partial a}=\int\frac{a}{\sqrt{a^2-x^2}}dx=a\sin^{-1}(x/a)$, as claimed.

Equivalently, one can just differentiate the integrand with respect to a, although without the justification above I'm not sure whether you'd have known that was allowed!

III. (a) $(e^{2y}-1)=x(e^{2y}+1)\Rightarrow(1-x)e^{2y}=1+x\Rightarrow e^{2y}=\frac{1+x}{1-x}\Rightarrow y=\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$.
(b) $c_2=c_1+\pi/2$ (try alternative substitution $x=\cos u$).
(c) $c_1=c_2$. To show equivalence, let $x=\cosh(y)\quad\Rightarrow e^{2y}-2xe^y+1=0\quad\Rightarrow e^y=x\pm\sqrt{x^2-1}\quad\Rightarrow y=\cosh^{-1}x=\ln(x\pm\sqrt{x^2-1})$ (and think about domain of $\cosh^{-1}x$).

IV. (a) $\frac{1}{3}\ln(5/2)$, (b) $\frac{1}{2}(1+e^{-\pi})$

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