INTEGRATION BY SUBSTITUTION AND PARTS

5 minute review. Recall the chain rule for differentiating a function of a function, namely $\frac{d}{dx}f(u(x)) = \frac{df}{du}\frac{du}{dx}$, then explain integration, running through these steps: (1) introduce a new variable $u = u(x)$ (or $x = x(u)$); (2) replace the measure using $dx = \frac{dx}{du}du$ (or $dx = \frac{1}{\frac{du}{dx}}du$); (3) integrate with respect to u; (4) replace u with x in the answer.

Recap how *partial fractions* can be used in integration, perhaps with the example $\int \frac{1}{1-x^2}\,dx$. Recall the product rule for differentiation, $\frac{d}{dx}(uv) = u'v + uv'$, and integrate and rearrange to obtain the *integration by parts* formula $\int uv'dx = uv - \int u'v\,dx$.

Class warm-up. Integrate (a) $\int x^2\exp(x^3)dx$, (b) $\int \frac{x^2}{1+x^3}dx$, (c) $\int \frac{4}{(x-1)(x+1)^2}\,dx$ (using partial fractions), (d) $\int x\ln x\,dx$ (using integration by parts).

Problems. (Choose from the below)

I. **Simple substitutions**. Find the following indefinite integrals by choosing a suitable substitution.

(a) $\displaystyle\int t\cos(t^2-1)\,dt$ (c) $\displaystyle\int \frac{x}{x^2+1}\,dx$

(b) $\displaystyle\int x\sqrt{1+x^2}\,dx$ (d) $\displaystyle\int \frac{\cos t}{\sqrt{1+\sin t}}\,dt$

II. **Trigonometric substitutions**. Find the indefinite integrals using the suggested substitutions.

(a) $\displaystyle\int \frac{1}{1+9x^2}\,dx, \quad 3x = \tan u$ (b) $\displaystyle\int \frac{1}{\sqrt{25-16x^2}}\,dx, \quad 4x = 5\sin u$

III. **Completing the square**. Find the following.

(a) $\displaystyle\int \frac{dx}{x^2+6x+10}$ (b) $\displaystyle\int \frac{dx}{\sqrt{3-2x-x^2}}$

IV. **Partial fractions**. Find the following indefinite integrals using the method of partial fractions as appropriate.

(a) $\displaystyle\int \frac{(x+1)}{x(x+3)}\,dx$ (c) $\displaystyle\int \frac{y\,dy}{y^3-y^2+y-1}$

(b) $\displaystyle\int \frac{dx}{(x^2+1)(x+1)}$ (d) $\displaystyle\int \frac{3x+3}{(x-1)^3(2x+1)}\,dx$

V. **Integration by parts**. Evaluate the following using integration by parts.

(a) $\displaystyle\int te^t\,dt$; (b) $\displaystyle\int \ln x\,dx$; (c) $\displaystyle\int y^3e^{-y^2}\,dy$; (d) $\displaystyle\int \cosh^{-1}u\,du$.

VI. **Log-of-a-log***. Use the substitution $x = e^u$ to show that

$$\int \frac{dx}{x\ln x} = \ln(\ln x) + c.$$

Let $I_n(x)$ be defined as follows.

$$I_1(x) = \int \frac{dx}{x\ln x},\ I_2(x) = \int \frac{dx}{x\ln(x)\ln(\ln x)},\ I_3(x) = \int \frac{dx}{x\ln(x)\ln(\ln x)\ln(\ln(\ln x))},\ldots$$

Note that, as n gets larger, x has to be very large and positive in order that its logarithm can be taken repeatedly.

Find $I_2(x)$ using integration by substitution. Show that $I_n(e^u) = I_{n-1}(u)$ (in other words, that substituting $x = e^u$ in $I_n(x)$ gives $I_{n-1}(u)$), and hence write down a general formula for $I_n(x)$. Check it is valid by differentiating.

For the warm-up: (a) $\int x^2 \exp(x^3) dx = \frac{1}{3}\exp(x^3)$, (b) $\int \frac{x^2}{1+x^3} dx = \frac{1}{3}\ln(1+x^3)$,
(c) $\frac{4}{(x-1)(x+1)^2} = \frac{1}{x-1} - \frac{1}{x+1} - \frac{2}{(x+1)^2}$, so $\int \frac{4\,dx}{(x-1)(x+1)^2} = \ln\left|\frac{x-1}{x+1}\right| + \frac{2}{x+1} + c$, and
(d) $\int x \ln x\, dx = \frac{1}{4}x^2\,(2\ln x - 1)$.

Selected answers and hints. (All answers should include a constant of integration.)

I. (a) $\frac{1}{2}\sin(t^2-1)$, (b) $\frac{1}{3}\left(1+x^2\right)^{3/2}$, (c) $\frac{1}{2}\ln(1+x^2)$, (d) $2\sqrt{1+\sin t}$.

II. (a) $\frac{1}{3}\arctan(3x)$, (b) $\frac{1}{4}\arcsin\left(\frac{4}{5}x\right)$.

III. (a) $\tan^{-1}(x+3)$, (b) $\sin^{-1}\left(\frac{1}{2}(x+1)\right)$.

IV. (a) $\frac{1}{3}\ln\left|x(x+3)^2\right|$, (b) $\frac{1}{2}\ln|x+1| - \frac{1}{4}\ln|1+x^2| + \frac{1}{2}\tan^{-1}(x)$ (c) $\frac{1}{2}\ln|y-1| - \frac{1}{4}\ln|1+y^2| + \frac{1}{2}\tan^{-1}(y)$, (d) $\frac{2}{9}\ln\left|\frac{x-1}{2x+1}\right| + \frac{1}{3(x-1)} - \frac{1}{(x-1)^2}$.

V. (a) $(t-1)e^t$, (b) $x(\ln x - 1)$ using $u = \ln x$, $v' = 1$, (c) $-\frac{1}{2}\left(y^2+1\right)\exp(-y^2)$, (d) $u\cosh^{-1}u - \sqrt{u^2-1}$.

VI. $I_n(e^u)$ is the result of making the substitution $x = e^u$ in $I_n(x)$. That is,

$$\begin{aligned} I_n(e^u) &= \int \frac{dx}{e^u \ln(e^u)\ldots\ln(\ln\ldots(\ln e^u)\ldots)} \\ &= \int \frac{\frac{dx}{du}du}{e^u u \ldots \ln(\ln\ldots(u)\ldots)} \\ &= \int \frac{e^u du}{e^u u \ldots \ln(\ln\ldots(u)\ldots)} \\ &= I_{n-1}(u). \end{aligned}$$

Putting $x = \ln u$ in the given identity, we get

$$I_n(x) = I_{n-1}(\ln x) = I_{n-2}(\ln(\ln x)) = \ldots$$

It follows that

$$I_n(x) = \underbrace{\ln(\ln(\ln\ldots(\ln(x))\ldots))}_{n+1\text{ times}} + c.$$

For more details, start a thread on the discussion board.

Extra Problems.

I. **Substitutions**. Find the following indefinite integrals by choosing a suitable substitution.

(a) $\int \sin^2 x \cos x \, dx$

(b) $\int (t+1)\sqrt{t^2+2t} \, dt$

(c) $\int \frac{3x}{x^2+a^2} \, dx$

(d) $\int \frac{\cosh u}{1+\sinh^2 u} \, du$

II. **Trigonometric substitutions**. Find the indefinite integrals using the suggested substitutions.

(a) $\int \tan x \, dx, \quad u = \cos x;$

(b) $\int \frac{1}{x^2+2x+5} \, dx, \quad x+1 = 2\tan u;$

(c) $\int \sec 2x \tan 2x \, dx, \quad u = \cos 2x.$

III. **Completing the square**. Find the following.

(a) $\int \frac{\cos x \, dx}{\sin^2 x + 2\sin x + 10}$

(b) $\int \frac{dx}{x^2+10x+29}$

IV. **Partial fractions**. Find the following indefinite integrals using the method of partial fractions as appropriate.

(a) $\int \frac{dx}{x^2+3x-10}$

(b) $\int \frac{dz}{(z-2)^2(z+1)}$

V. **Integration by parts**. Evaluate the following using integration by parts.

(a) $\int x^2 \cosh x \, dx$

(b) $\int x^n \ln x \, dx \quad (n \neq -1)$

(c) $\int \ln(t^2+a^2) \, dt$

(d) $\int \tan^{-1} u \, du$

VI. **Recurrence formulae***.

(a) Let $I_n = \int x^n e^{ax} dx$ where $n \geq 0$ is an integer and a is a (possibly complex) constant. Using integration by parts, show that, for $n > 0$,

$$I_n = \frac{1}{a}\left(x^n e^{ax} - nI_{n-1}\right)$$

and $I_0 = \frac{1}{a}e^{ax} + c$. Find I_1, I_2 and I_3. Show that

$$\frac{I_n}{n!} = \frac{1}{a}\left(\frac{x^n}{n!}e^{ax} - \frac{I_{n-1}}{(n-1)!}\right),$$

and find a general expression for I_n.

(b) Let $C_n = \int x^n \cos x \, dx$ and $S_n = \int x^n \sin x \, dx$. Show that, for $n > 0$,

$$\begin{aligned} C_n &= x^n \sin x - nS_{n-1}, \\ S_n &= -x^n \cos x + nC_{n-1}. \end{aligned}$$

and $C_0 = \sin x + c$, $S_0 = -\cos x + c$. Following a similar approach to above, can you find a general expression for C_n and S_n?

(c) When $a = i$, what's the relationship between I_n, S_n and C_n?

Selected answers and hints. (All answers should include a constant of integration.)

I. (a) $\frac{1}{3}\sin^3 x$, (b) $\frac{1}{3}(t^2+2t)^{3/2}$ (c) $\frac{3}{2}\ln\left(x^2+a^2\right)$.

II. (a) $\ln|\sec x|$, (b) $\frac{1}{2}\tan^{-1}\left(\frac{1}{2}(x+1)\right)$, (c) $\frac{1}{2}\sec(2x)$.

III. (a) $\frac{1}{3}\tan^{-1}\left(\frac{1}{3}(\sin x+1)\right)$ (b) $\frac{1}{2}\tan^{-1}\left(\frac{1}{2}(x+5)\right)$.

IV. (a) $\frac{1}{7}\ln\left|\frac{x-2}{x+5}\right|$, (b) $\frac{1}{9}\ln\left|\frac{z+1}{z-2}\right| - \frac{1}{3(z-2)}$.

V. (a) $(x^2+2)\sinh x - 2x\cosh x$, (b) $\frac{x^{n+1}}{(n+1)^2}\left((n+1)\ln x - 1\right)$,
(c) $t\ln(t^2+a^2) - 2t + 2a\tan^{-1}(t/a)$, (d) $u\tan^{-1}u - \frac{1}{2}\ln|1+u^2|$.

VI. (a) A general formula for I_n is

$$\begin{aligned} I_n &= n!\frac{e^{ax}}{a^{n+1}}\left(\frac{(ax)^n}{n!} - \frac{(ax)^{n-1}}{(n-1)!} + \frac{(ax)^{n-2}}{(n-2)!} - \ldots + (-1)^{n-1}ax + (-1)^n\right) + c \\ &= n!\frac{e^{ax}}{a^{n+1}}\sum_{k=0}^{n}\frac{(-1)^k(ax)^{n-k}}{(n-k)!} + c. \end{aligned}$$

(b) The general formulas are

$$\begin{aligned} C_n &= n!\left(\frac{x^n}{n!} - \frac{x^{n-2}}{(n-2)!} + \frac{x^{n-4}}{(n-4)!} - \ldots\right)\sin x \\ &\quad + n!\left(\frac{x^{n-1}}{(n-1)!} - \frac{x^{n-3}}{(n-3)!} + \frac{x^{n-5}}{(n-5)!} - \ldots\right)\cos x + c \end{aligned}$$

and

$$\begin{aligned} S_n &= n!\left(\frac{x^{n-1}}{(n-1)!} - \frac{x^{n-3}}{(n-3)!} + \frac{x^{n-5}}{(n-5)!} - \ldots\right)\sin x \\ &\quad - n!\left(\frac{x^n}{n!} - \frac{x^{n-2}}{(n-2)!} + \frac{x^{n-4}}{(n-4)!} - \ldots\right)\cos x + c \end{aligned}$$

where the series in each bracket terminates before the power of x involved becomes negative.

(c) Using Euler's relation, $e^{ix} = \cos x + i\sin x$, we find that $I_n = C_n + iS_n$.

For more details, start a thread on the discussion board.