INTEGRATION BY INSPECTION

5 minute review. Remind students that $F(x)$ is an *indefinite integral* of $f(x)$ if and only if $F'(x) = f(x)$, and that we write

$$\int f(x)\,dx = F(x) + c.$$

Here $f(x)$ is the *integrand* and dx is the *measure of integration*. The indefinite integral is only defined up to an additive constant since, for any indefinite integral $F(x)$, $\frac{d}{dx}(F+c) = \frac{dF}{dx} + \frac{d}{dx}(c) = f + 0 = f$. Describe the process of integration by inspection.

Class warm-up. "Every function that can be differentiated generates a function that can be integrated." (a) Find the derivative of $\exp(-x^3)$, and hence find $\int x^2 \exp(-x^3)\,dx$. (b) Find the derivatives of e^{-x} and xe^{-x}. By taking a linear combination of results, find $\int xe^{-x}\,dx$. (You could mention that (a) can be done with integration by substitution, and (b) with integration by parts, which will follow next week.)

Problems. (Choose from the below)

I **Integration by inspection**. By looking for functions F such that $F'(x) = f(x)$, find the indefinite integrals $\int f(x)\,dx$ for the following integrands.

(a) $x^2 + x^3$
(b) $(1+x)^2$
(c) $\dfrac{1}{x^2}$
(d) $\dfrac{1}{(x+1)^2}$
(e) $\dfrac{1}{(3x+2)^3}$
(f) e^{4x}
(g) $\sin 3x$
(h) $\cosh 4x$
(i) $\sec^2 x$
(j) $(1-x^2)/(1+x)$
(k) $\dfrac{1}{x}$ [see Q4]
(l) $\dfrac{1}{2x+1}$
(m) $\dfrac{x}{1+x}$

II **Integrating powers of trigonometric functions**. By using an appropriate double-angle formula and the identity $\sin^2 x + \cos^2 x = 1$, find

(a) $\int \cos^2 x\,dx$;

(b) $\int \sin^3 x\,dx$;

(c) $\int \sin^4 x\,dx$;

(d) Can you find $\int \cos^4 x\,dx$ using (c)? (Hint: replace x with $x + \pi/2$. It's possible to do this without using integration by substitution!)

III **Other integrands**. Let $F(x)$ be an indefinite integral of $f(x)$, where

$$f(x) = x(x-1)^2(x-2)^3.$$

(a) Find and classify the stationary points of $F(x)$.

(b) Sketch $F(x)$ in the region $0 \le x \le 2$, assuming that $F(0) = 0$.

IV **Logarithms***.

(a) Let $y = \ln(x)$, where $x > 0$. By taking exponentials of both sides and differentiating, show that $y' = \frac{1}{x}$. Hence write down the indefinite integral of $\frac{1}{x}$ for $x > 0$.

(b) Repeat with $y = \ln(-x)$, where $x < 0$.

(c) What's the best expression for $\int \frac{1}{x}\,dx$?

(d) Now repeat for $y = \ln(g(x))$ and $\ln(-g(x))$ for an unknown function $g(x)$. Hence find an expression for $\int \frac{g'(x)}{g(x)}\,dx$.

For the warm-up, $\int x^2 \exp(-x^3)\,dx = -\frac{1}{3}\exp(-x^3)$ and $\int xe^{-x}\,dx = -e^{-x}(x+1)$.

Selected answers and hints.

I. All answers should include a constant of integration, omitted here.

(a) $\frac{1}{3}x^3 + \frac{1}{4}x^4$;
(b) $\frac{1}{3}(1+x)^3$;
(c) $-x^{-1}$;
(d) $-(x+1)^{-1}$;
(e) $-\frac{1}{6(3x+2)^2}$;
(f) $\frac{1}{4}e^{4x}$;
(g) $-\frac{1}{3}\cos 3x$;
(h) $\frac{1}{4}\sinh 4x$;
(i) $\tan x$;
(j) $x - \frac{1}{2}x^2$;
(k) $\ln|x|$;
(l) $\frac{1}{2}\ln|2x+1|$;
(m) $x - \ln|1+x|$.

II. (a) $\frac{1}{2}x + \frac{1}{4}\sin(2x) + c$;

(b) $-\cos x + \frac{1}{3}\cos^3 x + c$ (or equivalents);

(c) $\int \sin^4 x dx = \frac{3}{8}x - \frac{1}{8}\left(3 + 2\sin^2 x\right)\cos x \sin x + c$.

(d) Let $F(x) = \frac{3}{8}x - \frac{1}{8}\left(3 + 2\sin^2 x\right)\cos x \sin x$ and let $u = x + \pi/2$. Then, using the fact that $\sin(x+\pi/2) = \cos x$ and $\cos(x+\pi/2) = -\sin x$,

$$\begin{aligned} F(u) &= \frac{3}{8}u - \frac{1}{8}\left(3 + 2\sin^2 u\right)\cos u \sin u \\ &= \frac{3}{8}x + \frac{3}{8}(\pi/2) - \frac{1}{8}\left(3 + 2\cos^2(x)\right)(-\sin(x)).\cos(x) \\ &= \frac{3}{8}x + \frac{1}{8}\left(3 + 2\cos^2 x\right)\sin x \cos x + \frac{3\pi}{16}. \end{aligned}$$

But $F(x)$ differentiates to give $\sin^4 x$, so differentiating the above we get

$$\frac{d}{dx}(F(u)) = \frac{d}{du}(F(u)).\frac{du}{dx} = F'(u).1 = \sin^4(u) = \cos^4 x.$$

It follows that the expression for $F(u)$ above is an indefinite integral of $\cos^4 x$; in other words, $\int \cos^4 x\,dx = \frac{3}{8}x + \frac{1}{8}\left(3 + 2\cos^2 x\right)\sin x \cos x + c$.

III. $F(x)$ has stationary points at $x = 0$, 1, and 2, which are maximum, inflexion and minimum, respectively. $F(x)$ passes through zero at $x = 0$. Hopefully that's enough info to draw a rough sketch.

IV. (a) $\int \frac{1}{x}\,dx$ seems to be $\ln(x) + C$.

(b) Now $\int \frac{1}{x}\,dx$ seems to be $\ln(-x) + C$.

(c) The above is best summarised by $\int \frac{1}{x}\,dx = \ln(|x|) + C$, as I'm sure you already know.

(d) Similarly, we find $\int \frac{g'(x)}{g(x)}\,dx = \ln(|g(x)|) + C$.

For more details, start a thread on the discussion board.

Extra Problems.

I **Differentiation**.

(a) Evaluate $\frac{d}{dx}\left(\frac{xe^x}{x+1}\right)$.

(b) Find $\frac{dy}{dx}$ given $x = 2t/(1+t^2)$ and $y = t^2/(1+t^2)$.

(c) Differentiate x^n from first principles, where $n > 1$ is an integer.

II **Series**.

(a) Find the first four non-zero terms of the Maclaurin series for $\sin(2x+2)$.

(b) Use l'Hopital's rule to find, $\lim_{x\to 0} \frac{\cos(x)-1}{x^2}$.

III **Complex numbers**.

(a) Find the real and imaginary parts of $z = (3+i)/(1+3i)$.

(b) Write down $z = 1 + i\sqrt{3}$ in its polar and exponential forms, and plot it on the Argand diagram.

(c) Use de Moivre's theorem to find $\sin(3\theta)$ in terms of powers of $\sin(\theta)$, and $\cos(3\theta)$ in terms of powers of $\cos(\theta)$.

IV **Vectors**.

(a) Find the constants α and β, given that the vector $\mathbf{r} = (\alpha, 4, \beta)$ is perpendicular to each of the vectors $\mathbf{a} = (1, 2, 3)$ and $\mathbf{b} = (-1, 1, 1)$.

(b) If $\mathbf{a} = (2, 1, 0)$, $\mathbf{b} = (3, 5, 2)$ and $\mathbf{c} = (1, 1, -1)$, verify that $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{a}.\mathbf{c})\mathbf{b} - (\mathbf{b}.\mathbf{c})\mathbf{a}$.

Selected answers and hints.

I. (a) The derivative is $e^x - (xe^x)/(x+1)^2$.

(b) $\frac{dy}{dx} = t/(1-t^2)$.

(c) $\frac{d}{dx}(x^n) = nx^{n-1}$, as well known. The working relies on the binomial expansion of $(x+h)^n$.

II. (a) $\sin(2x+2) = \sin(2) + 2x\cos(2) - 2x^2\sin(2) - (4/3)x^3\cos(2) + ...$

(b) The limit is -1/2.

III. (a) $\text{Re}(z) = 3/5$, $\text{Im}(z) = -4/5$.

(b) $z = 2(\cos(\frac{\pi}{3}) + i\sin(\frac{\pi}{3})) = 2e^{i\frac{\pi}{3}}$.

(c) By using the fact that $(\cos\theta + i\sin(\theta))^3 = \cos(3\theta) + i\sin(3\theta)$ and comparing real and imaginary parts, one finds that $\sin(3\theta) = 3\sin(\theta) - 4\sin^3(\theta)$ and $\cos(3\theta) = 4\cos^3(\theta) - 3\cos(\theta)$.

IV. (a) $\alpha = 1, \beta = -3$.

(b) $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (-3, 9, 6)$.

For more details, start a thread on the discussion board.